MULTIPLIER THEOREMS ON ANISOTROPIC HARDY SPACES

by

LI-AN DANIEL WANG

A DISSERTATION

Presented to the Department of Mathematics
and the Graduate School of the University of Oregon
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy

June 2012

DISSERTATION APPROVAL PAGE

Student: Li-An Daniel Wang

Title: Multiplier Theorems on Anisotropic Hardy Spaces

This dissertation has been accepted and approved in partial fulfillment of the requirements for the Doctor of Philosophy degree in the Department of Mathematics by:

Marcin Bownik	Chair
James Isenburg	Member
N. Chris Phillips	Member
Yuan Xu	Member
Van Kopen	Outside Member

and

Kimberly Andrews Espy	Vice President for Research & Innovation/ Dean of the Graduate School

Original approval signatures are on file with the University of Oregon Graduate School.

Degree awarded June 2012

DISSERTATION ABSTRACT

Li-An Daniel Wang

Doctor of Philosophy

Department of Mathematics

June 2012

Title: Multiplier Theorems on Anisotropic Hardy Spaces

We extend the theory of singular integral operators and multiplier theorems to the setting of anisotropic Hardy spaces. We first develop the theory of singular integral operators of convolution type in the anisotropic setting and provide a molecular decomposition on Hardy spaces that will help facilitate the study of these operators. We extend two multiplier theorems, the first by Taibleson and Weiss and the second by Baernstein and Sawyer, to the anisotropic setting. Lastly, we characterize the Fourier transforms of Hardy spaces and show that all multipliers are necessarily continuous.

CURRICULUM VITAE

NAME OF AUTHOR: Li-An Daniel Wang

GRADUATE AND UNDERGRADUATE SCHOOLS ATTENDED:
University of Oregon, Eugene, OR
University of British Columbia, Vancouver, BC, Canada

DEGREES AWARDED:
Doctor of Philosophy, 2012, University of Oregon
Master of Science, 2010, University of Oregon
Bachelor of Science, 2005, University of British Columbia

AREAS OF SPECIAL INTEREST:
Harmonic analysis, Fourier analysis, non-classical Hardy spaces, dog sitting

PROFESSIONAL EXPERIENCE:

Graduate Teaching Fellow, University of Oregon, 2005-2012

GRANTS, AWARDS AND HONORS:

Johnson Fellowship 2009, 2010

ACKNOWLEDGEMENTS

Many thanks to Professor Bownik for his guidance and patience, which is especially appreciated as I took my time entering such a classical and saturated field as harmonic analysis.

Many thanks to my committee members for their support, time, and understanding. I am especially indebted to Professor Phillips for his time and effort in providing feedback for this dissertation.

Many thanks to the Department of Mathematics for all the support, accommodations, encouragements, and opportunities. It has been a privilege representing the department as an instructor in the classroom, a task I carry out with great pride because of the examples set for us. This department is truly amazing, among all other UO departments and mathematics departments everywhere else.

Many thanks to my fellow graduate students for their company, wisdom, time, and friendship, especially those I entered the program with: John Jasper, Daniel Moseley, Kristy Pelatt, Jim Urick, and Daniel Westerman. Without them, I would not have made it past week 3 of fall term, 2005.

Many thanks to my family, for reacting to my career choice as if it is the most natural thing in the world.

Many thanks to all the baristas for accommodating my incessant requests of coffee, ice, water, water, water.

TABLE OF CONTENTS

Chapter Page

LIST OF TABLES

CHAPTER I

INTRODUCTION

The most fundamental space in harmonic analysis is the Lebesgue space $L^p(\mathbb{R}^n)$, defined for $p > 0$ by the collection of functions that are finite under the L^p-norm:

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p} < \infty.$$

For a function K that is locally integrable away from the origin, we define a singular integral operator by $Tf = K * f$. A fundamental problem is the study of singular integral operators on L^p. Denote $B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$. To facilitate the study of such operators, one uses the Hardy-Littlewood maximal operator, defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy.$$

As is well known, M is bounded $L^p \to L^p$ for $p > 1$.

However, for $p = 1$, M is only weakly bounded on L^1. If $E \subset \mathbb{R}^n$ is measurable, denote $|E|$ as the Lebesgue measure of E. Then there exists a constant $C = C(n, p)$ so that for all $\alpha > 0$ and $f \in L^1$,

$$|\{x \in \mathbb{R}^n : Mf(x) > \alpha\}| \leq C \frac{\|f\|_{L^1}}{\alpha}.$$

where $|E|$ denotes the Lebesgue measure of a set E. In fact, elementary considerations show that if $f \in L^1$ and $Mf \in L^1$, then f must be identically 0. Analogous to this, many singular integral operators are also bounded on L^p when $p > 1$, but only weakly bounded on L^1. Furthermore, when $p < 1$, the space L^p is even more pathological, as it may include non-integrable functions. One example is the function $f(x) = \frac{1}{x}\mathbf{1}_{[-1,1]}$, which is in $L^p(\mathbb{R})$ for any $p < 1$, but f itself is not locally integrable. The natural replacement turns out to be the Hardy space $H^p(\mathbb{R}^n)$.

1.1. Hardy Spaces

We now introduce Hardy spaces and explain why they are the natural spaces to consider when $p \leq 1$. Hardy spaces first originated in complex analysis, characterized as spaces of holomorphic functions on the unit disk or the upper half plane. But complex-analytic methods were not readily generalizable to higher dimensions. Through the ground breaking work of Stein and Weiss [SW68] and Fefferman and Stein [FS72], the theory of H^p spaces was reformulated in the setting of real analysis, thus opening it up to a broad range of new approaches. We will now briefly describe three such characterizations of H^p: The maximal characterization [FS72], the atomic characterizations [Coi74b], and the Littlewood Paley characterization [FJ90]. After the introduction of singular integrals, we will also briefly describe the singular integral characterization of H^p.

1.1.1. Maximal Characterization of H^p

Recall that the Schwartz class $S(\mathbb{R}^n)$ is the collection of functions $\phi \in C^\infty$ such that for all multi-indices α, β,

$$\|\phi\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta \phi(x)| < \infty.$$

We denote $S'(\mathbb{R}^n)$ as the dual space of S, which we call the class of tempered distributions. Let $\tilde{\phi}(y) = \phi(-y)$ be the reflection operator and $T_x\phi(y) = \phi(y - x)$

be the translation operator. If $f \in S'$ and $\phi \in S$, we use the bracket notation $\langle f, \phi \rangle$ to denote the distribution f is acting on ϕ. By defining the convolution $f * \phi$ by $(f * \phi)(x) = \langle f, T_x(\tilde{\phi}) \rangle$, we say $f \in S'(\mathbb{R}^n)$ is bounded if $f * \phi \in L^\infty$ whenever $\phi \in S$. Note that this definition is a generalization of the convolution between two functions in L^1.

With their origins in complex analysis, H^p spaces initially relied on Poisson integrals in their definition. If Γ is the classical Gamma function, denote $c_n = \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}}$. The Poisson kernel on $\mathbb{R}^n$ is defined by

$$P(x) = \frac{c_n}{(1 + |x|^2)^{\frac{n+1}{2}}}.$$

We define the (isotropic) dilation by $P_t(x) = t^{-n} P(x/t)$. With this kernel, we define the Poisson integral of $f \in S'$ as a function on the upper half plane $\mathbb{R}^{n+1}_+ = \{(x, t) : x \in \mathbb{R}^n, t > 0\}$:

$$u(x, t) = (f * P_t)(x),$$

and the non-tangential maximal function of u by

$$u^*(x) = \sup_{t>0} \sup_{y \in B(x,t)} |u(y, t)|.$$

Definition 1.1. Let $0 < p < \infty$. Suppose $f \in S'(\mathbb{R}^n)$ is a bounded distribution. Then $f \in H^p(\mathbb{R}^n)$ if $u^* \in L^p$.

This definition is restrictive, due to the use of the Poisson integral. It is the key observation in [FS72] that the Poisson kernel can be replaced by any non-trivial smooth function. If $\Phi \in S$, we define the radial maximal function by

$$M^0_\Phi f(x) = \sup_{t>0} |(f * \Phi_t)(x)|.$$

Now let $\mathcal{F} = \{\|\cdot\|_{\alpha_i,\beta_i}\}_{\alpha_i,\beta_i}$ be any finite collection of seminorms on S. We denote

$$S_\mathcal{F} = \{\Phi \in S : \|\Phi\|_{\alpha,\beta} \leq 1 \text{ for all } \|\cdot\|_{\alpha,\beta} \in \mathcal{F}\}.$$

Given such a collection $\mathcal{F}$ of seminorms, we define the grand maximal function associated with $\mathcal{F}$ by

$$M^0_\mathcal{F} f(x) = \sup_{\Phi \in S_\mathcal{F}} M^0_\Phi f(x).$$

The key connection between M_Φ and $M_\mathcal{F}$ is that the two are actually comparable in H^p-norm, made concrete by the following theorem of [FS72]. For norms on a space X, we denote $\|f\|_X \simeq \|g\|_X$ by the existence of two constants, independent of f and g, so that

$$c_1\|f\|_X \leq \|g\|_X \leq c_2\|f\|_X.$$

If $\Phi \in S$, we define the isotropic dilation by $\Phi_t(x) = t^{-n}\Phi(x/t)$, analogous to the Poisson kernel. The following theorem is originally from Fefferman and Stein [FS72]. The present form is taken from Chapter 3, Theorem 1 of Stein [Ste93].

Theorem 1.1. *Let $p \in (0,\infty]$ and $f \in S'$. Then the following conditions are equivalent:*

1. *There is $\Phi \in S$ with $\int \Phi \neq 0$ so that $M^0_\Phi f \in L^p(\mathbb{R}^n)$.*

2. *There is a finite collection $\mathcal{F}$ of seminorms so that $M^0_\mathcal{F} f \in L^p(\mathbb{R}^n)$.*

3. *The distribution f is bounded and $u^* \in L^p(\mathbb{R}^n)$.*

With this theorem, we say $f \in H^p$ if any one of the above three conditions holds, and for a fixed Φ with $\int \Phi dx \neq 0$, we have $\|f\|_{H^p} \simeq \|M^0_{\mathcal{F}}\|_{L^p} \simeq \|M^0_\Phi f\|_{L^p} \simeq \|u^*\|_{L^p}$, and any one of these can serve as the Hardy space norm. Furthermore, H^p generalizes L^p as follows: When $p > 1$, $H^p = L^p$. If $p = 1$, $H^1 \subset L^1$. And when $p < 1$, H^p and L^p are not compatible, as there exist $f \in H^p$ that are not even functions, and any non-locally integrable functions in L^p will not be in S', and therefore will not be in H^p. Here is an example illustrating this difference.

Example 1.1. Let $x \in \mathbb{R}^n$. The Dirac-delta distribution δ_x is defined by $\delta_x(\varphi) = \varphi(x)$ for $\varphi \in S$. Let $f = \delta_{-1} - \delta_1 \in S'(\mathbb{R})$. We will show $f \in H^p(\mathbb{R})$ for $p \in (1/2, 1)$. To see this, we need to show that there exists $\varphi \in S(\mathbb{R})$, $\int \varphi \neq 0$, such that $M^0_\varphi f \in L^p(\mathbb{R})$. We let $\varphi \in S$ that is compactly supported:

$$\operatorname{supp}(\varphi) = [-2, 2], \qquad \text{and} \qquad \varphi\Big|_{[-1,1]} = 1.$$

We denote $C(\varphi) = \max\{\|\varphi\|_\infty, \|\varphi'\|_\infty\}$. The maximal operator is then given by:

$$M^0_\varphi f(x) = \sup_{t>0} \left| \frac{1}{t} \left(\varphi\left(\frac{x-1}{t}\right) - \varphi\left(\frac{x+1}{t}\right)\right)\right|.$$

Fix $x \in \mathbb{R}$. Since the support of φ is in $[-2, 2]$, we have

$$\begin{aligned} \varphi\left(\frac{x-1}{t}\right) &= 0 \text{ if } t < \frac{|x-1|}{2} \\ \varphi\left(\frac{x+1}{t}\right) &= 0 \text{ if } t < \frac{|x+1|}{2}. \end{aligned}$$

Therefore, in estimating $M^0_\varphi f(x)$, we only need to consider $t \geq \frac{|x-1|}{2}$ or $t \geq \frac{|x+1|}{2}$.

1. Now suppose $|x| > 2$. Then for each t, by the Mean Value Theorem, we have $\xi = \xi(x,t) \in (\frac{x-1}{t}, \frac{x+1}{t})$ such that:

$$\left|\frac{2}{t}\left(\varphi\left(\frac{x-1}{t}\right) - \varphi\left(\frac{x+1}{t}\right)\right)\right| = \frac{2}{t^2}\left|t\left(\varphi\left(\frac{x-1}{t}\right) - \varphi\left(\frac{x+1}{t}\right)\right)\right| = \frac{2}{t^2}|\varphi'(\xi)|.$$

Let $C_\varphi = \|\varphi\|_\infty$. With our prior restriction on t, we then have:

$$\sup_{t>0} |(\varphi_t * f)(x)| \leq \sup_{t \geq \max(\frac{|x-1|}{2}, \frac{|x+1|}{2})} \frac{1}{t^2}|\varphi'(\xi)| \leq \begin{cases} \frac{C_\varphi}{|x-1|^2} & \text{if } x < 0 \\ \frac{C_\varphi}{|x+1|^2} & \text{if } x > 0. \end{cases}$$

We conclude that if $p > 1/2$, then

$$\int_{|x|>2} |M_\varphi f(x)|^p dx < \infty.$$

So the "tail" of $M_\varphi f$ is taken care of by the fact that φ is compactly supported and $\varphi \in C^1(\mathbb{R})$.

2. Now suppose $|x| \leq 2$. Then since φ is bounded, we have:

$$\sup_{t>0} |\varphi_t * f(x)| \leq \begin{cases} \frac{C_\varphi}{|x+1|} & \text{if } -2 \leq x \leq 0 \\ \frac{C_\varphi}{|x-1|} & \text{if } 0 \leq x \leq 2. \end{cases}$$

We conclude that if $p \in (0,1)$, then

$$\int_{[-2,2]} |M_\varphi f(x)|^p dx \leq \int_{-2}^{0} \frac{C_\varphi}{|x+1|^p} dx + \int_0^2 \frac{C_\varphi}{|x-1|^p} dx < \infty.$$

3. Now we show $f \notin H^p$ for $p \geq 1$. To do this, we make one small restriction on φ by requiring $\varphi(-1) \neq 0$. (In light of the maximal characterization, such a change will not affect whether $f \in H^p$). Given x, set $t = |x-1|$. We then have:

$$M_\varphi f(x) \geq \frac{1}{|x-1|} \cdot \left| \varphi\left(\frac{x-1}{|x-1|}\right) - \varphi\left(\frac{x+1}{|x-1|}\right) \right| = \frac{1}{|x-1|} \cdot \left| \varphi\left(-1\right) - \varphi\left(\frac{x+1}{|x-1|}\right) \right|.$$

Our goal is to 'do away' with one of the φ-terms above so the expression above is approximately $\frac{1}{|x-1|}$ for x near 1. To do this, we first note that as $x \to 1^-$, $\frac{x+1}{|x-1|} \to \infty$. Formally, for every $\epsilon > 0$, there exists $\delta > 0$ such that $|x-1| < \delta$ implies

$$\left| \varphi\left(\frac{x+1}{|x-1|}\right) \right| < \epsilon.$$

Making sure ϵ is small enough, we can find a $\delta > 0$ such that whenever $1 - \delta < x < 1$,

$$M_\varphi f(x) \geq \frac{1}{|x-1|} \cdot |(|\varphi((-1)\,| - \epsilon)| = \frac{C_\varphi}{|x-1|}.$$

Therefore we have

$$\int_{1-\delta}^{1} M_\varphi f(x) dx \geq \int_{1-\delta}^{1} \frac{C_\varphi}{|x-1|} dx = \infty.$$

This shows $f \notin H^1(\mathbb{R})$, or any L^p for $p \geq 1$.

4. Now to see why we require $p > 1/2$, we will look at the long-term behavior of $M_\varphi f(x)$. It suffices to show there exists one $\varphi \in S$ for which $M_\varphi f \notin L^p$. With our previous φ, we make one more requirement, namely, that $\varphi'(1) > 0$. Then by the Mean Value Theorem, for a fixed x and t, there exists $\xi \in (\frac{x-1}{t}, \frac{x+1}{t})$

such that

$$\frac{2}{t}\left|\varphi\left(\frac{x-1}{t}\right)-\varphi\left(\frac{x+1}{t}\right)\right|=\frac{2}{t^2}\varphi'(\xi).$$

We set $t=|x-1|$. Then as $x\to\infty$, the interval ξ is from shrinks:

$$\xi\in\left(\frac{x-1}{|x-1|},\frac{x+1}{|x-1|}\right)=\left(1,\frac{x+1}{|x-1|}\right)\to\{1\}.$$

Since φ' is continuous at 1, if $\epsilon>0$ is small enough, then there exists $N>0$ such that whenever $x>N$, $\xi\in(1,\frac{x+1}{|x-1|})$ implies $|\varphi'(\xi)-\varphi'(1)|<\epsilon$, or $|\varphi'(\xi)|>\varphi'(1)-\epsilon>0$. All together, for such an ϵ and N, and $x>N$, we have

$$M_\varphi f(x)\geq\frac{C_\varphi}{|x-1|^2}.$$

Therefore if $2p\leq 1$, or $p\leq 1/2$, we have

$$\int_N^\infty(M_\varphi f(x))^p dx\geq\int_N^\infty\frac{C_\varphi}{|x-1|^{2p}}dx=\infty.$$

This shows $f\notin H^p(\mathbb{R})$ for any $p<1/2$.

Of course, it is not surprising that $f\notin L^p$ since elements in L^p are defined almost everywhere, while the Dirac-delta masses are concentrated at two points. This immediately rules out the possibility that f be represented by locally integrable functions.

1.1.2. Littlewood-Paley Characterization

The Littlewood-Paley characterization of H^p provides another unifying view on why H^p naturally replaces L^p when $p\leq 1$. Furthermore, this generalization also unites Sobolev and Lipschitz spaces, as shown by the work of Frazier and Jawerth

[FJ90], though it follows the work of many others in both classical harmonic analysis and wavelet theory.

We briefly describe the H^p-L^p connection. Let $\Psi \in S(\mathbb{R}^n)$ be radial with Fourier transform $\hat{\Psi}$ is supported in the annulus $\frac{1}{2} + \frac{1}{10} \leq |\xi| \leq 2 - \frac{1}{10}$, and for all $\xi \neq 0$,

$$\sum_{j\in\mathbb{Z}} \hat{\Psi}(2^{-j}\xi) = 1.$$

Associated with this bump function, we define

$$\Delta_j(f) = \Psi_{2^{-j}} * f.$$

Then following characterization of H^p spaces is originally due to Frazier and Jawerth [FJ90]. Its present form is taken from Grafakos [Gra09], Section 6.4.6.

Theorem 1.2. *Let Ψ and Δ_j be as above, and let $0 < p \leq 1$. Then there exists a constant $C = C(n, p, \Psi)$ so that for all $f \in H^p(\mathbb{R}^n)$,*

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\Delta_j(f)|^2 \right)^{1/2} \right\|_{L^p} \leq C\|f\|_{H^p}.$$

Conversely, suppose a tempered distribution f satisfies

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\Delta_j(f)|^2 \right)^{1/2} \right\|_{L^p} < \infty.$$

Then there exists a unique polynomial Q such that $f - Q \in H^p$, satisfying the estimate

$$\|f - Q\|_{H^p} \leq C \left\| \left(\sum_{j\in\mathbb{Z}} |\Delta_j(f)|^2 \right)^{1/2} \right\|_{L^p}.$$

This characterization is generalized in Triebel-Lizorkin spaces. If $\alpha \in \mathbb{R}$ and $0 < p, q \leq \infty$ and $f \in S'$, we say $f \in \dot{F}_p^{\alpha,q}$

$$\|f\|_{\dot{F}_p^{\alpha,q}} = \left\| \left(\sum_{j\in\mathbb{Z}} (2^{j\alpha} |\Delta_j(f)|)^q \right)^{1/q} \right\|_{L^p} < \infty.$$

In light of this, we have

$$\dot{F}_p^{0,2} = \begin{cases} L^p & \text{if } 1 < p < \infty, \\ H^p & \text{if } 0 < p \leq 1. \end{cases}$$

In other words, under the Littlewood-Paley decomposition, H^p and L^p are unified under one definition, as is the case of the maximal characterization. This again shows H^p is the natural replacement for L^p when $p \leq 1$.

1.1.3. Atomic Decomposition of H^p

The next characterization of H^p, through the work of Coifman [Coi74b] and Latter [Lat78], uses a special type of functions called atoms that provides a decomposition of distributions in H^p. We denote the floor of x by $\lfloor x \rfloor$, defined as the largest integer k such that $k \leq x$. Let $p \in (0, 1]$ and $1 \leq q \leq \infty$, satisfying $p < q$. A function a is a (p, q) atom on $\mathbb{R}^n$ if a is supported on a ball B, and satisfies

1. The size condition: $\|a\|_{L^q} \leq |B|^{\frac{1}{q} - \frac{1}{p}}$
2. The vanishing moment condition: For all $|\alpha| \leq N = \left\lfloor n(\frac{1}{p} - 1) \right\rfloor$, we have $\displaystyle\int a(x)\, x^\alpha dx = 0$.

If $q = \infty$, the size condition takes the form $\|a\|_\infty \leq |B|^{-1/p}$.

Remark 1.1. We now explain the importance of both conditions. For simplicity, suppose $a(x)$ is a $(p,2)$ atom.

1. The exponent of $|B|$ is negative, so the larger the support of $a(x)$, the smaller its L^2-norm, and vice versa. This is a crucial fact if we consider the previous example of $f = \delta_{-1} - \delta_1 \in S$. Each Dirac-delta mass is, essentially, an approximation of identity at the points $x = -1$ and $x = 1$. Classically, we can duplicate the Dirac-delta mass at $x = 0$ by taking two smooth functions $f, \varphi \in S$ with $\int_{\mathbb{R}^n} \varphi \, dx = 1$. Then it is well-known that

$$\lim_{\epsilon \to 0} f * \varphi_\epsilon = f \quad \text{everywhere.}$$

   In particular, as $\epsilon \to 0$, the support of $\varphi_\epsilon(x) = \epsilon^{-1}\varphi(x/\epsilon)$ is shrinking, while its L^∞ norm is going to ∞, exhibiting the same behavior as atoms. This is not surprising, as we expect the atoms to be building blocks of H^p.

2. The vanishing moment condition is also crucial. As we will see later when we estimate $\hat{f}$, a a necessary condition for $f \in H^p$ is that $\hat{f}(0) = 0$. In fact, we require $|\hat{f}(\xi)| \leq C|x|^{n(\frac{1}{p}-1)}$, so the order of 0 at the origin necessitates the vanishing moment conditions.

The atomic decomposition of $H^p(\mathbb{R})$ is originally due to Coifman [Coi74b]. The case of $H^p(\mathbb{R}^n)$ is due to Latter [Lat78]. The following form of the atomic decomposition is from Stein [Ste93], Chapter 3, Theorem 2.

Theorem 1.3. *Let $p \in (0,1]$ and $1 \leq q \leq \infty$, satisfying $p < q$. A distribution $f \in S'$ is in $H^p(\mathbb{R}^n)$ exactly when there exists a sequence $\{\lambda_j\} \in \ell^p$ and (p,q) atoms $\{a_j\}$ so* *that*

$$f = \sum_{j=1}^{\infty} \lambda_j a_j$$

with the convergence in H^p, i.e.,

$$\lim_{N\to\infty} \left\| f - \sum_{j=1}^{N} \lambda_j a_j \right\|_{H^p} = 0.$$

Furthermore, we have

$$\|f\|_{H^p} \simeq \inf \left\{ \left(\sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p} : f = \sum_{j=1}^{\infty} \lambda_j a_j \right\},$$

with the infimum taken over all possible decompositions of f.

The most direct proof of the atomic decomposition uses (p,∞) atoms. By showing every (p,∞) atom is also a (p,q) atom, it is easy to show that the decomposition holds for (p,q) atoms for any p and q. Because the decomposition into (p,∞) atoms is the most accessible, in much of the literature, to prove the boundedness of an operator $T\colon H^p \to Y$ on a normed space Y, one first proves that for all (p,∞) atoms a,

$$\|Ta\|_Y \leq C, \tag{1.1}$$

with C independent of the atom, and extends this estimate to all $f \in H^p$ as follows. With $f = \sum_j \lambda_j a_j$ decomposed using (p,∞) atoms, one wishes the following were true:

$$\|Tf\|_{H^p}^p = \left\| \sum_j \lambda_j T(a_j) \right\|_{H^p}^p \leq \sum_j |\lambda_j|^p \|Ta_j\|_{H^p}^p \leq C \sum_j |\lambda_j|^p \leq C\|f\|_{H^p}^p.$$

However, the first equality above does not hold in general since the sum is infinite. In fact, Bownik provided an example in [Bow05] of an operator that is uniformly bounded on (p, ∞) atoms but is not bounded on H^p.

To overcome this, we observe that for any q such that $1 \leq q \leq \infty$ (and $p < q$ if $p = 1$), the collection of finite combinations of (p, q) atoms, $H^{p,q}_{fin}$, with the norm

$$\|f\|_{H^{p,q}_{\text{fin}}} = \inf \left\{ \left(\sum_{j=1}^{N} |\lambda_j|^p \right)^{1/p} : f = \sum_j \lambda_j a_j \right\},$$

is dense in H^p. Here, the infimum taken over all possible finite decompositions. Meda, Sjögren, and Vallarino [MSV08] showed that if $q < \infty$, then $\|f\|_{H^{p,q}_{fin}} \simeq \|f\|_{H^p}$. Then to prove $T : H^p \to Y$ is bounded, it suffices to show $T : H^{p,q}_{\text{fin}} \to Y$ is bounded. With each $f \in H^{p,q}_{fin}$ admitting a finite atomic decomposition, we can pass T directly through the summation, and indeed, it suffices to prove (1.1) for (p, q) atoms. The equivalence $\|f\|_{H^{p,q}_{fin}} \simeq \|f\|_{H^p}$ for $q < \infty$ has been extended to a number of settings, most notably the weighted anisotropic setting in [BLYZ08], which also includes our anisotropic setting. For most of our results on boundedness of operators, we will prove (1.1) for $(p, 2)$ atoms and $Y = L^p$ or H^p.

1.2. Singular Integral Operators

Having defined Hardy spaces, we now give a general introduction to singular integral operators. Given a set $E \in \mathbb{R}^n$, we denote $E^c = \mathbb{R}^n \backslash E$ as the complement of E.

Definition 1.2. Suppose $K \in S'$. Then the operator $Tf(x) = (K * f)(x)$, initially defined on the Schwartz class $S(\mathbb{R}^n)$, is a singular integral operator if $\hat{K} \in L^\infty(\mathbb{R}^n)$ and K coincides with a function $k \in L^1_{\text{loc}}(\mathbb{R}^n \backslash \{0\})$, there exists a constant C_K such that such that for all $y \in \mathbb{R}^n$,

$$\int_{B(0,2|y|)^c} |K(x-y) - K(x)| dx \leq C_K. \tag{1.2}$$

Condition (1.2) is called Hörmander's condition.

Remark 1.2. We briefly mention the consequences of the definition of a singular integral operator. Details can be found in Chapter II.5 of [GCRdF85].

1. If $K \in S'$ and $f \in S(\mathbb{R}^n)$, the convolution $K * f \in C^\infty$ has, at most, polynomial growth.

2. By requiring $\hat{K} \in L^\infty$, we can extend T to be a bounded operator on $L^2(\mathbb{R}^n)$, with $\|T\|_{L^2 \to L^2} = \|\hat{K}\|_{L^\infty}$.

3. Hörmander's condition is the key ingredient in showing T is also weakly bounded on $L^1(\mathbb{R}^n)$: There is a constant C such that for all $\alpha > 0$,

$$|\{x : |Tf(x)| > \alpha\}| \leq C \frac{\|f\|_{L^1}}{\alpha}.$$

4. With $T : L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$ weakly bounded and $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ (strongly) bounded, we can apply an interpolation result to obtain boundedness of $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ for $p \in (1, 2)$. A duality argument then gives boundedness for $p \in (2, \infty)$.

When $p \leq 1$, we can replace L^p by the Hardy space H^p, on which the singular integral operator can be expected to be bounded. In particular, for $p = 1$, $T : H^1(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ is bounded. However, we will need additional conditions on T if $T : H^1(\mathbb{R}^n) \to H^1(\mathbb{R}^n)$ is to be bounded. This is not surprising, as $H^1(\mathbb{R}^n)$

is a subspace of $L^1(\mathbb{R}^n)$, so showing $Tf \in H^1$ for $f \in H^1$ will require additional assumptions on K. We summarize what we have so far with Table 1.1.

TABLE 1.1. Singular Integral Operators I

	$+\ \hat{K} \in L^\infty$	+ Hörmander
$T : L^2 \to L^2$	Bounded	Bounded
$T : L^1 \to L^{1,\infty}$		Weakly Bounded
$T : L^p \to L^p$, for $p \in (1, \infty)$		Bounded
$T : H^1 \to L^1$		Bounded

We now address the following questions.

Q1: Under what additional conditions on K will T be bounded from H^1 to H^1?

Q2: If $p \in (0, 1)$, under what additional conditions on K will T be bounded from H^p to L^p and to H^p?

The answers to these two questions are *regularity conditions.* With additional regularity on K, we will indeed have boundedness $H^p \to L^p$ for values of p not too close to 0. The closer we want the values of p to be 0, the more regularity will be required. We now state the regularity conditions.

Definition 1.3. Suppose $Tf = K * f$ is a singular integral operator.

1. We say T is regular if the function $K \in L^1_{loc}(\mathbb{R}^n \backslash \{0\})$ satisfies

$$|K(x)| \leq B|x|^{-n} \quad \text{for} \quad x \in \mathbb{R}^n \backslash \{0\} \tag{1.3}$$

$$|K(x - y) - K(x)| \leq B|y|\ |x|^{-(n+1)} \quad \text{for } |x| > 2|y| > 0. \tag{1.4}$$

2. If T is regular, we say T has additional regularity of order m if the kernel K is in $C^{m+1}(\mathbb{R}^n \backslash \{0\})$, and there exists C such that for all multi-index β such that

$|\beta| \leq m+1$ and $x \neq 0$, we have

$$|\partial^\beta K(x)| \leq C|x|^{-(n+|\beta|)}. \tag{1.5}$$

The constant C is independent of x and β.

The following theorem is taken from [GCRdF85], Chapter III.7.

Theorem 1.4. *Suppose T is a singular integral operator, with kernel K.*

1. *If T is regular, then $T : H^p \to L^p$ is bounded if $\frac{n}{n+1} < p \leq 1$.*
2. *If T has additional regularity of order m, (i.e., T satisfies* (1.3), (1.4), *and* (1.5)*), then $T : H^p \to L^p$ is bounded if $\frac{n}{n+m+1} < p \leq 1$. Furthermore, $T : H^p \to H^p$ is also bounded for the same values of p.*

Classically, the proof of each part of the theorem starts by proving $\|Ta\|_{L^p}$ and $\|Ta\|_{H^p}$ are uniformly bounded for (p, ∞) atoms. The second, more involved part, is to show the operator T can be passed through the infinite atomic decomposition because of the regularity condition. But in light of the finite atomic decomposition, it suffices to prove the uniform boundedness on atoms. Table 1.2 incorporates the new information.

TABLE 1.2. Singular Integral Operators II

Range of p	Spaces	$K \in S'$ $\hat{K} \in L^\infty$	With Hörmander	With Regularity	+ Additional k- Regularity
$p = 2$ $p = 1$ $1 < p < \infty$	$L^2 \to L^2$ $L^1 \to L^{1,\infty}$ $L^p \to L^p$	Bounded	Bounded Weakly Bdd Bounded	Bounded Weakly Bdd Bounded	Bounded Weakly Bdd Bounded
$\frac{n}{n+1} < p \leq 1$	$H^p \to L^p$			Bounded	Bounded
$\frac{n}{n+m+1} < p \leq 1$ $\frac{n}{n+m+1} < p \leq 1$	$H^p \to L^p$ $H^p \to H^p$				Bounded Bounded

1.2.1. Singular Integral Characterization of H^p

We now provide one more characterization of H^p using singular integral operators. Classically, the Hilbert transform $Hf = \text{p.v.} \displaystyle\int \frac{f(y)}{x-y} dy$ is known to be weakly bounded on $L^1(\mathbb{R})$. The Hardy space $H^1(\mathbb{R})$ was defined to be all $f \in L^1(\mathbb{R})$ such that $Hf \in L^1(\mathbb{R})$. This differs from the action of the Hardy-Littlewood maximal operator M, for which $Mf \in L^1$ only if $f = 0$. For $n \geq 2$, $H^1(\mathbb{R}^n)$ can be formulated accordingly by generalizing the Hilbert transform using the Riesz transforms R_j. If $c_n = \Gamma(\frac{n+1}{2})/\pi^{\frac{n+1}{2}}$, the Riesz transforms are defined by

$$R_j(f)(x) = c_n \text{ p.v.} \int \frac{x_j - y_j}{|x-y|^{n+1}} f(y) dy,$$

These are natural generalizations of the Hilbert transform in n dimensions. The following result is due to Stein and Weiss [SW68]. Its present form is taken from Grafakos [Gra09], Section 6.7.4.

Theorem 1.5. *For $n \geq 2$, there exists a constant C_n such that for $f \in L^1(\mathbb{R}^n)$, we have*

$$C_n \|f\|_{H^1} \leq \|f\|_{L^1} + \sum_{k=1}^{n} \|R_k(f)\|_{L^1}.$$

When $n = 1$,

$$C_1 \|f\|_{H^1} \leq \|f\|_{L^1} + \|H(f)\|_{L^1},$$

for all $f \in L^1$.

This is a very involved result that uses the Poisson kernel extensively. Given $f \in L^1$, we define $n+1$ functions

$$\begin{aligned} u_1(x,t) &= (P_t * R_1(f))(x), \\ &\vdots \\ u_n(x,t) &= (P_t * R_n(f))(x), \\ u_{n+1}(x,t) &= (P_t * f)(x). \end{aligned}$$

This system satisfies the generalized Cauchy-Riemann equations:

$$\sum_{j=1}^{n+1} \frac{\partial u_j}{\partial x_j} = 0, \qquad \frac{\partial u_j}{\partial x_k} - \frac{\partial u_k}{\partial x_j} = 0 \text{ for } k, j \in \{1, \dots, n+1\} \text{ with } k \neq j.$$

By defining $F = (u_1, \dots, u_{n+1})$, the function

$$|F|^q = \left(\sum_{j=1}^{n+1} |u_j|^2 \right)^{q/2}$$

is subharmonic when $q \geq (n-1)/n$, that is, $\Delta(|F|^q) \geq 0$ on $\mathbb{R}^{n+1}_+$. This property, called harmonic majorization, is the key step in the singular integral characterization of H^1. The formulation of $H^p(\mathbb{R}^n)$ can also be achieved with singular integrals, and is used in [BS85] to extend multiplier theorems from $H^p \to L^p$ to $H^p \to H^p$.

1.3. Multiplier Theorems

Singular integral operators can be studied from the frequency side, resulting in multiplier operators. Results of these nature, which do not start with conditions on the kernel K, are called multiplier theorems. Instead, the starting point is conditions on $m = \hat{K}$ where $\hat{K}$ denotes the Fourier transform of K. If $K \in L^1(\mathbb{R}^n)$, then the transform is given by the usual formula

$$\hat{K}(\xi) = \int_{\mathbb{R}^n} K(x) e^{-2\pi i x \xi} dx.$$

If $K \in S'$, then its Fourier transform is defined by duality: For $\phi \in S$,

$$\langle \hat{K}, \phi \rangle = \langle K, \hat{\phi} \rangle.$$

will give a bounded operator on appropriate pairs of spaces. To start, we define what a multiplier operator is.

Definition 1.4. Let $m \in L^\infty(\mathbb{R}^n)$. The multiplier operator T_m is defined initially on $S(\mathbb{R}^n)$ by the formula

$$T_m f = (\hat{f} m)^\vee.$$

Given that $m \in L^\infty$, we can initially define the operator $T_m : S \to C_0 \cap L^2$, or $T_m : L^2 \to L^2$, and aim to extend T_m to $H^p \to H^p$. There are two approaches. The first approach, of Taibleson and Weiss [TW80], uses *molecules*, which are a generalization of atoms, to decompose H^p spaces. The definition of a molecule can be reformulated in the frequency domain, where most of the analysis takes place, using the Fourier transform. The second approach, of Baernstein and Sawyer [BS85], generalizes the notion of a molecule to that of a Herz space, whose norm dominates the L^p-norm. This initially gives boundedness $H^p \to L^p$, and is improved to boundedness $H^p \to H^p$ using the Riesz characterization of H^p. We now state these two classical results in more detail.

1.3.1. Multiplier Theorem: Molecular Approach

We introduce molecules and provide a molecular decomposition of $H^p(\mathbb{R}^n)$. Unlike atoms, they do not have to be compactly supported.

Definition 1.5. We say (p, q, b) is admissible if $0 < p \leq 1 < q \leq \infty$ and $b > \frac{1}{p} - \frac{1}{q}$. We define $\theta = (\frac{1}{p} - \frac{1}{q})/b$, so that $\theta \in (0, 1)$. A function $M \in L^2(\mathbb{R}^n)$ is a (p, q, b) molecule centered at $x_0 \in \mathbb{R}^n$ if it satisfies the size condition

$$N(M) = \|M\|_q^{1-\theta} \; \||x - x_0|^{nb} M\|_q^{\theta} < \infty,$$

and the vanishing moment condition: for all multiindex β such that $|\beta| \leq \lfloor n(\frac{1}{p}-1) \rfloor = N$,

$$\int_{\mathbb{R}^n} x^\beta M(x) dx = 0.$$

We call $N(M)$ the molecular norm of M.

Roughly speaking, we think of atoms as the basic building blocks of Hardy spaces, and singular integral operators map atoms to molecules, due to the fact that T, as a convolution operator, 'spreads' an atom a out so it is no longer compactly supported. Fortunately, these molecules are still specialized enough to decompose H^p in a meaningful way.

Remark 1.3. Here are two immediate connections between atoms and molecules.

1. The molecular norms of atoms are uniformly bounded: For every admissible (p, q, b), there exists a constant C such that every (p, q) atom is a (p, q, b) molecule whose molecular norm is uniformly bounded $N(a) \leq C$.

2. T maps atoms to molecules: Suppose T is a singular integral operator with additional regularity of order m, $\frac{n}{n+m+1} < p \leq 1 < q < \infty$, and (p, q, b) is admissible. Then T maps (p, q) atoms to (p, q, b) molecules, and there is a constant C such that

$$N(Ta) \leq C.$$

The following molecular decomposition is due to Taibleson and Weiss [Tai66]. The present form is taken from [GCRdF85], Chapter III.7.

Theorem 1.6. *Let (p, q, b) be admissible, and M be a (p, q, b) molecule in $\mathbb{R}^n$. Then $M \in H^p(\mathbb{R}^n)$ and there exists a constant C independent of M such that*

$$\|M\|_{H^p(\mathbb{R}^n)} \leq CN(M).$$

Furthermore, a tempered distribution f is in $H^p(\mathbb{R}^n)$ exactly when, as tempered distributions, there are (p, q, b) molecules $\{M_j\}_j$ such that

$$f = \sum_{j=1}^{\infty} M_j,$$

and $\sum_j N(M_j)^p < \infty$. In particular, there exists a constant C such that with the above decomposition,

$$\|f\|_{H^p}^p \leq C \sum_j N(M_j)^p \text{ and } \sum_j N(M_j)^p \leq C\|f\|_{H^p}^p,$$

with C independent of f and the decomposition.

The first benefit of this molecular decomposition is that $N(M)$, as a product of L^q norms, is much easier to compute than $\|M\|_{H^p}$. Second, when $q = 2$, the definition of a molecule can be reformulated using the Fourier transform. Using Parseval's identity, we have the following characterization. Let $k \in \mathbb{N}$. Then $F \in L^2$ is the Fourier transform of a $(p, 2, k/n)$ molecule M exactly when

$$\sup_{|\alpha|=k} \|F\|_2^{1-\theta} \; \|\partial^\alpha F\|_2^\theta < \infty,$$

$$\partial^\beta F(0) = 0 \;\; \text{for every } \beta \text{ such that } \; 0 \leq |\beta| \leq N.$$

These are the frequency analogues of the size and vanishing moment conditions in the definition of a molecule. The pointwise condition on $\partial^\beta F$ makes sense because the vanishing moment condition on M is equivalent to the existence of enough integrable weak derivatives so that $\partial^\beta F$ is continuous, and be defined pointwise.

We now present the multiplier theorem of Taibleson and Weiss [TW80]. The key condition is the Hörmander condition. The present form of the theorem is from [GCRdF85], Chapter III.7.

Definition 1.6. Let $k \in \mathbb{N}$ and $m \in L^1_{\mathrm{loc}}(\mathbb{R}^n \backslash \{0\})$. We say m satisfies the Hörmander condition of order k if there exists a constant A such that for all indices β such that $|\beta| \leq k$,

$$\sup_{R>0} \int_{R<|x|<2R} |\partial^\beta m(x)|^2 dx \leq A R^{n-2|\beta|}. \tag{1.6}$$

Theorem 1.7. *Let $n \in \mathbb{N}$ and $k \in \mathbb{N}, k > n/2$. Suppose m satisfies Hörmander's condition of order k, and A the constant in* (1.6)*. Then there exists a constant C such that if a is a $(p, 2, k-1)$ atom centered at the origin with $0 < p \leq 1$ and $\frac{1}{p} - \frac{1}{2} < k/n$,* *then* $(m\hat{a})^{\vee}$ *is a* $(p,2,k/n)$*-molecule centered at the origin with*

$$N((m\hat{a})^{\vee}) \leq CA,$$

with $C = C(p,k,n)$*. Furthermore, for every* p *such that* $1/(\frac{k}{n}+\frac{1}{2}) < p \leq 1$, $T_m : H^p(\mathbb{R}^n) \to H^p(\mathbb{R}^n)$ *is bounded, with*

$$\|Tf\|_{H^p} = \|(m\hat{f})^{\vee}\|_{H^p} \leq CA\|f\|_{H^p}.$$

To prove this theorem, we perform our analysis in the frequency domain: though our goal is to show $N((m\hat{a})^{\vee}) \leq CA$, we can equivalently show there exists another general constant C, independent of f, such that

$$\sup_{|\alpha|=k} \|m\hat{a}\|_2^{1-\theta}\ \|\partial^{\alpha}(m\hat{a})\|_2^{\theta} \leq C,$$
$$\partial^{\beta}(m\hat{a})(0) = 0 \quad \text{for every } 0 \ \leq |\beta| \leq m+1.$$

The second condition is immediate, since $\hat{a}(0) = 0$ due to its vanishing moment condition (in time). The first condition follows from combining the integral Hörmander's condition (1.6) on m with estimates on $\hat{a}$. Since $a(x)$ is compactly supported, we expect $\hat{a}$ to be infinitely differentiable, with certain decay properties. Specifically, the following are true. Denote $N = \lfloor n(\frac{1}{p}-1)\rfloor$. Given a $(p,2,k)$ atom a with $k \leq N$:

1. There exists a constant A of the form $A(k,n) = d(\frac{k+1}{n}+\frac{1}{2}) - 1 > 0$ such that for every α such that $0 \leq |\alpha| \leq k$,

$$|\partial^{\alpha}\hat{a}(\xi)| \leq C|\xi|^{k+1-|\alpha|}\ \|a\|_2^{-A(k,n)}.$$

2. There exists a constant B of the form $B(\alpha, r, n) = d(\frac{2|\alpha|}{n} + \frac{1}{r}) - 2$ such that for every index α and every $r \in [1, \infty]$, and $\frac{1}{r} + \frac{1}{r'} = 0$,

$$\| \, |\partial^\alpha \hat{a}|^2 \|_{r'} \leq C \|a\|_2^{-B(\alpha,r,n)}.$$

These two estimates are proved using the Fourier inversion formula and exploiting the vanishing moments condition with Taylor's approximation. They also lead to the important result of Taibleson and Weiss [TW80] that if $f \in H^p$, then $\hat{f}$ is continuous, and

$$|\hat{f}(\xi)| \leq C \|f\|_{H^p} \, |\xi|^{n(\frac{1}{p}-1)}. \tag{1.7}$$

Combining these estimates on $\hat{a}$ and condition (1.6) will give uniform boundedness of the molecular norm of $(m\hat{a})^\vee$. Since the molecular norm dominates the H^p norm, this conclude the first multiplier theorem.

Lastly, these estimates on $\hat{a}$ and (1.7) are extended in our anisotropic setting in Chapter IV.

1.3.2. Multiplier Theorem: Herz Spaces

The second result, from [BS85], generalizes the notion of a molecule using the Herz space. We define $\mathcal{A}_k = \{x \in \mathbb{R}^n : 2^k \leq |x| \leq 2^{k+1}\}$, the k^{th} dyadic annulus.

Definition 1.7. Suppose $1 \leq a \leq \infty$, $0 \leq \alpha < \infty$, and $0 < b \leq \infty$.

1. $\dot{K}_a^{\alpha,b}$ consists of all functions $f \in L^a_{\text{loc}}(\mathbb{R}^n \backslash \{0\})$ such that

$$\|f\|_{\dot{K}_a^{\alpha,b}} = \left(\sum_{k=-\infty}^{\infty} \left(\int_{\mathcal{A}_k} |f(x)|^a dx \right)^{b/a} 2^{k\alpha b} \right)^{1/b} < \infty.$$

2. We define $K_a^{\alpha,b} = L^a \cap \dot{K}_a^{\alpha,b}$, with

$$\|f\|_{K_a^{\alpha,b}} = \|f\|_{L^a} + \|f\|_{\dot{K}_a^{\alpha,b}}.$$

3. $B_a^{\alpha,b}$ consists of all functions $F \in L^a(\mathbb{R}^n)$ such that if $u(x,t)$ is the Poisson extension of f to $\mathbb{R}^n \times [0,\infty)$,

$$\|F\|_{B_a^{\alpha}} = \|F\|_{L^a} + \left(\int_0^\infty \left[t^{s-\alpha}(\int_{\mathbb{R}^n} |\partial_t^s u(\xi,t)|^a d\xi)^{1/a}\right]^b \frac{dt}{t}\right)^{1/b} < \infty,$$

Immediately, one sees that a (p,q,b) molecule is in $K_q^{\alpha,q}$ for some $\alpha > n\left(\frac{1}{p} - \frac{1}{q}\right)$. Second, by a simple application of Hölder's inequality, we have

$$\|f\|_{L^p} \leq C\|f\|_{\dot{K}_1^{n(\frac{1}{p}-1),p}}.$$

By the Fourier transform and Parseval's identity, we have the following facts.

Theorem 1.8. *Let $\eta \in C_c^\infty(\mathbb{R}^n)$ be supported on the annulus $\{\frac{1}{4} \leq |\xi| \leq 4\}$, takes values in $[0,1]$, and be identically 1 on the annulus $\{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\}$. Let $m \in L^1_{\text{loc}}(\mathbb{R}^n \backslash \{0\})$. For $\delta > 0$, define*

$$m_\delta(\xi) = m(\delta\xi)\eta(\xi).$$

1. Let $s \in \mathbb{N}$. Then m satisfies Hörmander's condition of order s exactly when

$$\sup_{\delta>0} \|(m_\delta)^\wedge\|_{K_2^{s,2}} < \infty.$$

2. *The Fourier transform $\mathcal{F}$ maps $B_2^{\alpha,q}$ to $K_2^{\alpha,q}$ isomorphically for $\alpha \geq 0$ and $0 < q \leq \infty$.*

We are now ready to state the multiplier theorem of [BS85]. There are two cases: when $p < 1$ and $p = 1$. From now on, we fix η and the notation m_δ from the previous theorem. The following two multiplier theorems are Theorem 3a and 3b of Baernstein and Sawyer [BS85], respectively.

Theorem 1.9 (Multiplier Theorem for $p < 1$)**.** *Suppose $p \in (0, 1)$, and*

$$M = \sup_{\delta>0} \|(m_\delta)^\wedge\|_{K_1^{n(\frac{1}{p}-1),p}} < \infty.$$

1. *There exists a constant $C = C(n, p)$ such that $T_m : H^p \to H^p$ is bounded, with $\|T_m\| \leq CM$.*

2. *Suppose that either $\alpha = n(\frac{1}{p} - \frac{1}{2})$ and $q \leq p$ or $\alpha > n(\frac{1}{p} - \frac{1}{2})$. If*

$$\sup_{\delta>0} \|(m_\delta)^\wedge\|_{K_2^{\alpha,q}} < \infty \qquad \textit{or equivalently} \qquad \sup_{\delta>0} \|m_\delta\|_{B_2^{\alpha,q}} < \infty,$$

then $T_m : H^p \to H^p$ is bounded.

Theorem 1.10 (Multiplier Theorem for $p = 1$)**.** *Let M be as in the previous theorem. Suppose $w : \mathbb{N} \cup \{0\} \to [1, \infty)$ is increasing: $1 \leq w(k) \leq w(k + 1) < \infty$ for all k, and define the weighted Herz space to be the set of all $f \in L^1(\mathbb{R}^n)$ such that*

$$\|f\|_{K(w)} = \int_{|x|<1} |f(x)|dx + \sum_{k=0}^{\infty} \left(\int_{\mathcal{A}_k} |f(x)|dx \right) w(k) < \infty.$$

If w also satisfies

$$W = \sum_{k=0}^{\infty} \frac{1}{w(k)^2} < \infty,$$

then there exists a constant $C = C(n)$ *such that* $T_m : H^1 \to H^1$ *is bounded, with* $\|T\|_{op} \leq CMW$.

The proofs of both results take place in the time domain, that is, they use the condition on $(m_\delta)^\wedge$ as a starting point. We will prove a uniform bound on Ta for all $(p, 2)$ atoms a. The fact that the Herz norm majorizes the L^p-norm means it suffices to show

$$\|Ta\|_{\dot{K}_1^{n(\frac{1}{p}-1),p}} \leq C,$$

which then yields the boundedness of $T : H^p \to L^p$. This result is improved to $T : H^p \to H^p$ by using the Riesz characterization of H^p, to show that for each Riesz transform R_j,

$$\|R_j(Ta)\|_{L^p} \leq C.$$

This is equivalent to $\|Ta\|_{H^p} \leq C$, and boundedness is proved for the first theorem (when $p < 1$). For the case $p = 1$, we bypass the Herz space and prove directly, for $(1, 2)$ atoms a, the uniform bound

$$\|Ta\|_{L^1} \leq C.$$

Both theorems start with a Littlewood-Paley decomposition of $(m\hat{a})^\vee$. Given η from Theorem 1.8, we define $\psi \in C_c^\infty$, with $\text{supp}(\psi) \subset \{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\}$, with values in $[0, 1]$ such that

$$\sum_{j \in \mathbb{Z}} \psi(2^{-j}\xi) = 1.$$

We define $\hat{a}_\delta(\xi) = (\hat{a})_\delta(\xi) = \hat{a}(\delta\xi)\ \psi(\xi)$, and define the following functions:

$$f_j = (m_{2^j})^\vee, \qquad b_j = (\hat{a}_{2^j})^\vee.$$

Then we have the decomposition:

$$(m\hat{a})^{\vee} = \sum_{j\in\mathbb{Z}} 2^{nj}(f_j * b_j)(2^j x).$$

The conditions on m take care of f_j. So the proof is mainly concerned with the analysis of b_j, as provided by the following two lemmas.

Lemma 1.1. *Let $N = \lfloor n(\frac{1}{p} - 1)\rfloor$ and suppose we have a $(p, 2)$ atom a supported on $B(0, 1)$, the unit ball, and $r > 0$ be a fixed constant. Then there exists a constant C, depending only on n, p, r, such that for all $j \le 0$,*

1. $|b_j(x)| \le C2^{j(N+1)}(1 + |x|)^{-r}$.
2. *For $j \ge 0$,*

$$|b_j(x)| \le \begin{cases} C2^{-jn} & \text{for all } x, \\ C|x|^{-r} & \text{for all } x \text{ such that } |x| > 2^{j+1}. \end{cases}$$

The constants C depend only on p, n, r.

Lemma 1.2. *Suppose $0 < p < 1$, $j \ge 0$ and $r > n/p$. Then there is a constant C, depending only on n, p, r, such that for all $g \in K_1^{n(\frac{1}{p}-1),p}$ and $Q \in L^1(\mathbb{R}^n)$ satisfying*

$$\int_{\mathbb{R}^n} |Q|dx \le 1 \qquad |Q(x)| \le |x|^{-r} \text{ if } |x| > 2^{j+1},$$

we have

$$\sum_{k=j+2}^{\infty} \left(\int_{\mathcal{A}_k} |g * Q|dx\right)^p 2^{kn(1-p)} \le C\|g\|^p_{K_1^{n(\frac{1}{p}-1),p}}.$$

These two lemmas are key ingredients in the multiplier theorems. In Chapter III, we give anisotropic adaptations of these lemmas. Finally, we remark that in [BS85], only (p, ∞) atoms are used in proving the uniform bound on $\|Ta\|_{H^p}$. In light of the work of Bownik in [Bow05], this is not sufficient. So in our work below, we employ $(p, 2)$ atoms in our results. This does not affect the estimates in any major way.

1.4. Anisotropic Structure on $\mathbb{R}^n$

Having introduced Hardy spaces, singular integral operators, and multiplier theorems in the isotropic setting, we now introduce the anisotropic setting in detail. The anisotropic structure considered here was motivated by wavelet theory, and replaces the Euclidean norm with a more general quasinorm assocaited with a dilation matrix. It is certainly not the first generalization of the underlying $\mathbb{R}^n$ structure. Calderón and Torchinsky [CT75, CT77] studied the parabolic setting, using dilations of continuous groups $\{A_t\}_{t>0}$ on $\mathbb{R}^n$, which arose naturally from the study of singular integral operators along curves. Folland and Stein [FS82] replaced the underlying space $\mathbb{R}^n$ with homogeneous groups, and Coifman and Weiss initiated the study of Hardy spaces on spaces of homogeneous type in [CW77]. However, the extension of (4.1) was not considered in the parabolic setting, and the Fourier transform takes a more abstract form on homogeneous groups. Moreover, the Fourier transform is not even considered on spaces of homogeneous type, as these spaces might not have an underlying group structure.

The key difference is in replacing the Euclidean norm with a more general anisotropic norm associated with a dilation matrix. For more details, see Bownik [Bow03].

Definition 1.8. Fix $n \in \mathbb{N}$. Let A be an $n \times n$ matrix, and define $b = |\det A|$.

1. A is a dilation matrix if all eigenvalues λ of A satisfy $|\lambda| > 1$.

2. A homogeneous quasinorm associated with a dilation A is a measurable mapping $\rho : \mathbb{R}^n \to [0, \infty)$ so that

   (a) $\rho(x) = 0$ exactly when $x = 0$,

   (b) $\rho(Ax) = b\rho(x)$ for all $x \in \mathbb{R}^n$,

   (c) there is a doubling constant $c > 0$ so that for all $x, y \in \mathbb{R}^n$, $\rho(x + y) \leq c(\rho(x) + \rho(y))$.

Now let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A, taken according to multiplicity, and ordered so that

$$1 < |\lambda_1| \leq \cdots \leq |\lambda_n|.$$

We choose λ_- and λ_+ such that

$$1 < \lambda_- < |\lambda_1| \leq |\lambda_n| < \lambda_+.$$

Then there exists $c' > 0$ so that for all $x \in \mathbb{R}^n$,

$$\frac{1}{c'}\lambda_-^j|x| \leq |A^j x| \leq c'\lambda_+^j|x| \text{ if } j \geq 0$$
$$\frac{1}{c'}\lambda_+^j|x| \leq |A^j x| \leq c'\lambda_-^j|x| \text{ if } j \leq 0.$$

If λ is an eigenvalue and A does not have Jordan blocks corresponding to any eigenvalue λ, then we can set $\lambda_- = |\lambda_1|$ and $\lambda_+ = |\lambda_n|$. We call the ratios $\zeta_\pm = \frac{\log \lambda_\pm}{\log b}$ the eccentricities of A. Roughly speaking, the larger the eccentricities are, the more A differs from the isotropic setting.

In the isotropic case, the 'basic' geometric object is the ball $B(x, r)$, centered at $x \in \mathbb{R}^n$ with radius r. This has the nice property that whenever $r_1 < r_2$, we have $B(x, r_1) \subset B(x, r_2)$. But for a general dilation matrix A, we do not expect $B(x, r) \subset A(B(x, r))$. Instead, if A is a dilation, then there exists an ellipsoid Δ and $r > 1$ such that

$$\Delta \subset r\Delta \subset A\Delta.$$

Scaling Δ so that it has measure 1, we define $B_k = A^k\Delta$ for all $k \in \mathbb{Z}$. Then we have

$$B_k \subset B_{k+1} \text{ and } \qquad |B_k| = b^k.$$

These nested ellipsoids will serve as the basic geometric object in the anisotropic setting. Conveniently, any two quasinorms ρ_1, ρ_2 associated with a dilation A are equivalent, that is, if there exists constants c_1 and c_2 so that for all $x \in \mathbb{R}^n$,

$$c_1\rho_1(x) \leq \rho_2(x) \leq c_2\rho(x),$$

then the Hardy spaces generated by the two quasinorms are exactly the same. Therefore it suffices to fix one particular quasinorm. We will use the following canonical quasinorm used throughout this dissertation. For $j, k \in \mathbb{Z}$ with $j \leq k$, we denote the annulus by $B_j \backslash B_k = \{x \in \mathbb{R}^n : x \in B_j, x \notin B_k\}$.

Definition 1.9. Let A be a dilation on $\mathbb{R}^n$ and $\{B_k\}_{k\in\mathbb{Z}}$ be the nested ellipsoids fixed above. The step homogeneous quasinorm ρ on $\mathbb{R}^n$ induced by the dilation A is

$$\rho_A(x) = \begin{cases} b^j & \text{if } x \in B_{j+1} \backslash B_j \\ 0 & \text{if } x = 0. \end{cases} \tag{1.8}$$

With this step norm, we define ω as the smallest integer so that $2B_0 \subset A^\omega B_0 = B_\omega$. Then this norm is a homogeneous quasinorm associated with the dilation A, with

$c = b^\omega$ as the doubling constant. We make the following observations, which are the analogues of what is commonly known in the isotropic case.

Proposition 1.1. *Let A be a dilation matrix.*

1. *Let $x, y \in \mathbb{R}^n$, with $\rho(x) = b^i$ and $\rho(y) = b^j$ for some $i, j \in \mathbb{Z}$. Then*

$$\rho(x+y) \leq b^\omega(\rho(x) + \rho(y)) \tag{1.9}$$

$$\rho(x-y) \geq b^{-\omega}\rho(x) - \rho(y). \tag{1.10}$$

2. *If for some $i \in \mathbb{Z}$ and $x, y \in B_i$, then $x + y \in B_{i+\omega}$.*

3. *If for some $i \in \mathbb{Z}$ and $x \notin B_{i+\omega}$ and $y \in B_i$, then $x + y \notin B_i$.*

The following lemma, which allows us to relate the Euclidean norm to the quasinorm, is due to Lemarie-Rieusset [LR94],.

Lemma 1.3. *Suppose ρ is a homogeneous quasinorm associated with dilation A. Define $\zeta_\pm = \dfrac{\log \lambda_\pm}{\log b}$. Then there is a constant c_A depending only on A such that:*

$$\frac{1}{c_A}\rho(x)^{\zeta_-} \leq |x| \leq c_A \rho(x)^{\zeta_+} \quad \textit{if} \quad \rho(x) \geq 1, \tag{1.11}$$

$$\frac{1}{c_A}\rho(x)^{\zeta_+} \leq |x| \leq c_A \rho(x)^{\zeta_-} \quad \textit{if} \quad \rho(x) < 1. \tag{1.12}$$

1.5. Anisotropic Hardy Spaces

Now that we have the anisotropic structure on $\mathbb{R}^n$, we can use this to develop the anisotropic Hardy spaces $H^p_A(\mathbb{R}^n)$ associated with the dilation matrix A.

Let S be the Schwartz class. In light of (1.11) and (1.12), an equivalent condition for $\varphi \in C^\infty(\mathbb{R}^n)$ to be in S is that for every multi-index α and integer $m \geq 0$,

$$\|\varphi\|_{\alpha,m} = \sup_{x\in\mathbb{R}^n} \rho(x)^m |\partial^\alpha \varphi(x)| < \infty.$$

In other words, replacing the Euclidean norm by the quasinorm will not change the space. Recall that the dual space of S, that is, the space of tempered distributions on $\mathbb{R}^n$, is denoted by $S'(\mathbb{R}^n)$. For $N \geq 0$, we also define the family

$$S_N = \{\varphi \in S : \|\varphi\|_{\alpha,m} \leq 1 \text{ for } |\alpha|, m \leq N\}.$$

For $\varphi \in S$ and $k \in \mathbb{Z}$, the anisotropic dilation is defined by $\varphi_k(x) = b^k \varphi(A^k x)$.

To define the anisotropic Hardy spaces, we need the following anisotropic versions of maximal operators.

Definition 1.10. Let $\varphi \in S$ satisfy $\int \varphi = 1$ and let $f \in S'$.

1. The radial maximal function of f respect to φ is defined as

$$M_\varphi^0 f(x) = \sup_{k\in\mathbb{Z}} |(f * \varphi_k)(x)|. \tag{1.13}$$

2. The grand maximal function of f is

$$M_N^0 f(x) = \sup_{\varphi\in S_N} M_\varphi^0 f(x). \tag{1.14}$$

With these definitions, we can define the anisotropic Hardy space, this time independent of the Poisson kernel.

Definition 1.11. For a given dilation A and $p \in (0, \infty)$, denote

$$N_p = \begin{cases} \lfloor (\frac{1}{p} - 1) \log b / \log \lambda_- \rfloor + 2 & \text{if } 0 < p \leq 1, \\ 2 & \text{if } p > 1. \end{cases}$$

We define the anisotropic Hardy space associated with the dilation A as

$$H^p_A(\mathbb{R}^n) = \{f \in S' : M_{N_p} f \in L^p\},$$

with the quasinorm $\|f\|_{H^p} = \|M_{N_p} f\|_p$.

As before, this maximal characterization is independent of the operator or test function used. Furthermore, for all $N \geq N_p$, we obtain the same Hardy space. The following is Theorem 7 of Bownik [Bow05].

Theorem 1.11. *Let $p \in (0, \infty)$ and suppose $\varphi \in S$ with $\int \varphi \neq 0$. Then for any $f \in S'$ and any $N \geq N_p$, the following are equivalent.*

1. $f \in H^p_A(\mathbb{R}^n)$
2. $M^0_N f \in L^p(\mathbb{R}^n$.

If so, then we have a constant, independent of f, such that

$$\|f\|_{H^p_A} = \|M^0_N f\|_p \leq C \|M^0_\varphi f\|_p.$$

With this maximal characterization in place, we give the atomic decomposition of H^p_A. We first define the anisotropic atoms, and then state the anisotropic version of theorem 1.3.

Definition 1.12. A triple (p,q,s) is admissible with respect to the dilation A if $0 < p \leq 1$, $1 \leq q \leq \infty$, and $p < q, s \in \mathbb{N}$, and $s \geq \lfloor (\frac{1}{p} - 1) \log b / \log \lambda_- \rfloor$. A (p,q,s) atom is a function $a(x)$ supported on an ellipsoid $x_0 + B_j$ for some $x_0 \in \mathbb{R}^n$ and $j \in \mathbb{Z}$, such that

1. (Size Condition) $\|a\|_q \leq |B_j|^{\frac{1}{q} - \frac{1}{p}}$,
2. (Vanishing Moments) For all α such that $|\alpha| \leq s$, we have $\displaystyle\int_{\mathbb{R}^n} a(x) x^\alpha dx = 0$.

The anisotropic atomic decomposition, the result of a number of lemmas, is found in Chapter 6 of [Bow05].

Theorem 1.12 (Atomic decomposition of H^p_A)**.** *Suppose (p,q,s) is admissible. Then $f \in H^p_A$ exactly when there exist $\{\lambda_j\} \in \ell^p$ and (p,q,s) atoms $\{a_j\}$ so that*

$$f = \sum_{j=1}^{\infty} \lambda_j a_j,$$

with convergence in H^p_A-norm. Furthermore we have

$$\|f\|_{H^p} \simeq \inf \left\{ \left(\sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p} : f = \sum_{j=1}^{\infty} \lambda_j a_j \right\},$$

where the infimum is taken over all possible decompositions of f.

The anisotropic setting, roughly speaking, lacks the rotational invariance property of the isotropic setting. This is reflected in a number of places, which we will discuss briefly below. In the isotropic setting on $\mathbb{R}^n$, the 'basic' geometric object is the n-dimensional ball. For example, the Hardy-Littlewood maximal operator is defined by taking supremum over all balls centered at the same point. Similarly, in the theory of real Hardy spaces, the non-tangential maximal function is defined by taking the supremum over a cone $M_\varphi f(x) = \sup_{t>0} |(f * \varphi_t)(x)|$ where $\varphi_t(y) = t^{-n}\varphi(y/t)$. We notice the dilation on φ, given by $y \mapsto \frac{y}{t}$ for $t > 0$. This operation scales $y \in \mathbb{R}^n$ evenly in all coordinates by the same scalar t. We can generalize this dilation as follows. Suppose $\lambda_1, \ldots, \lambda_n > 0$ are fixed. Then we define the dilation on $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ by

$$(y_1, \ldots, y_n) \mapsto (\lambda_1^t y_1, \ldots, \lambda_n^t y_n).$$

Though the dilation is still using only one parameter, t, the scaling in each coordinate is different. The basic geometric object, reflecting this dilation, is now an ellipsoid, which is not rotationally invariant. This dilation can be further generalized to that of a *dilation matrix*, which our anisotropic setting is based on.

Furthermore, the Euclidean norm $|x|$ itself is also rotationally invariant. So to appropriately study a general dilation and carry out our analysis, we not only have to use ellipsoids, but to also use a quasinorm $\rho(x)$ that behaves well with respect to the ellipsoids. Therefore any analysis that uses $|x|$ and its properties will now have to be modified. Lastly, we note that if A has large eccentricity, that is, $\frac{\log b}{\log \lambda_-} >> 0$, then the atoms will require additional vanishing moments compared to the isotropic setting. Table 1.3 summarizes these differences.

TABLE 1.3. Isotropic and Anisotropic Settings

	Isotropic	Anisotropic
Geometric object Norm	$B(x,r)$ Continuous: $\|x\|$	$\{B_j\}_{j\in\mathbb{Z}}$ Discrete: $\rho(x)$
Dilation on $\mathbb{R}^n$ Dilation on ϕ	$y \mapsto y/t$ $\phi_t(x) = t^{-n}\phi(x/t)$ for $t > 0$	$y \mapsto Ay$ $\phi_k(x) = b^k\phi(A^k x)$ for $k \in \mathbb{Z}$
Maximal Operator $M_\phi f(x)$ H^p	$\sup_{t>0} \|(f * \phi_t)(x)\|$ $M_\phi f \in L^p$	$\sup_{k\in\mathbb{Z}} \|(f * \phi_k)(x)\|$ $M_\phi f \in L^p$
Atoms Vanishing moments	Supported on balls $\|\alpha\| \le \lfloor n(\frac{1}{p} - 1)\rfloor$	Supported on ellipsoids $\|\alpha\| \le \lfloor(\frac{1}{p} - 1)\rfloor \log b/\log\lambda_-$

CHAPTER II

SINGULAR INTEGRAL OPERATORS AND MOLECULES

We now study anisotropic singular integral operators associated with a dilation matrix A, extending the classical formulation in Definition (1.2). For the rest of the section, we fix a dilation matrix A and its associated step quasinorm ρ as discussed in (1.8). Generally when extending the classical theory, global results involving norms go through, while local results involving pointwise estimates and algebraic identities break down and require more work.

2.1. Definition of Singular Integral Operators

If $u \in \mathbb{R}^n$, we denote $E_u = \{z : \rho(z) \geq b^{2\omega}\rho(u)\}$. Recall the complement E_u^c is defined by $\{x \in \mathbb{R}^n : x \notin E_u\}$.

Definition 2.1. Let $K \in S'$ satisfy the following properties:

$$\hat{K} \in L^\infty(\mathbb{R}^n), \tag{2.1}$$

and there exists a positive constant C_H such that for all $u \in \mathbb{R}^n \backslash \{0\}$,

$$\int_{E_u^c} |k(z-u) - k(z)| dz \leq C_H, \tag{2.2}$$

Then the operator $Tf = f * K$, defined initially on $S(\mathbb{R}^n)$, is called an anisotropic singular integral operator with kernel K.

Remark 2.1. Suppose T is an anisotropic singular integral operator associated with the kernel $K \in S'(\mathbb{R}^n)$. Then by (2.1), T extends to the operator $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, with operator norm $\|T\|_{L^2} = \|\hat{K}\|_\infty$. To see this, we use Parseval's identity:

$$\|Tf\|_2 = \|f * K\|_2 = \|\widehat{f * K}\|_2 = \|\hat{f} \cdot \hat{K}\|_2 \leq \|\hat{K}\|_\infty \|\hat{f}\|_2.$$

This is, of course, identical to the isotropic setting.

We will refer to (2.2) as the (anisotropic) Hörmander's condition, which can be strengthened to the following Lipschitz condition that is generally easier to check.

Definition 2.2. We say $k : \mathbb{R}^n \to \mathbb{C}$ satisfies the Lipschitz condition if there exist constants $C_L, \gamma > 0$ such that whenever $\rho(z) \geq \rho(u) b^{2\omega} > 0$,

$$|k(z-u) - k(z)| \leq C_L \rho(u)^\gamma \rho(z)^{-(1+\gamma)}. \tag{2.3}$$

Proposition 2.1. *If k satisfies the Lipshcitz condition, then it also satisfies Hörmander's condition* (2.2) *with C_H depending only on A, γ, C_L.*

Proof. If $u \in \mathbb{R}^n \backslash \{0\}$, then there exists a unique $j \in \mathbb{Z}$ such that $u \in B_{j+1} \backslash B_j$ and $\rho(u) = b^j$. Therefore

$$z \in E_u \Rightarrow z \notin B_{j+2\omega} \Rightarrow z \in \mathbb{R}^n \backslash B_{j+2\omega}.$$

Then

$$
\begin{aligned}
\int_{E_u^c} |k(z-u) - k(z)|dz &\le C_L \rho(u)^\gamma \int_{E_u^c} \rho(z)^{-1+\gamma} dz \\
&\le C_L \rho(u)^\gamma \sum_{m=j+2\omega}^{\infty} \int_{B_{m+1}\setminus B_m} \rho(z)^{-(1+\gamma)} dz \\
&= C_L \rho(u)^\gamma \sum_{m=j+2\omega}^{\infty} b^{-m(1+\gamma)}(b^{m+1} - b^m) \\
&= C_L (b-1) b^{j\gamma} \sum_{m=0}^{\infty} b^{-\gamma(m+j+2\omega)} \\
&= \left(\frac{(b-1)b^{-2\gamma\omega}}{1-b^{-\gamma}} \right) C_L = C_H,
\end{aligned}
$$

with C_H depending only on the constants C_L, γ, and A. $\square$

We will focus on singular integral operators of the form $Tf = K * f$, where the convolution is given by a convolution defined as a principal value type. Specifically, K will be represented by a locally integrable function k (denoted by lower case) on $\mathbb{R}^n \setminus \{0\}$ with operator action defined by

$$
Tf = K * f = \text{p.v.} \int k(x-y) f(y) dy.
$$

In this case, we write $K = \text{p.v.}k$, and will refer to T and K interchangeably. If k is a locally integrable function, and $K = \text{p.v.}k$, we seek sufficient conditions so that the operator $Tf = f * K$ is a singular integral operator. In particular,

Q1 Under what conditions on the function k will $K = \text{p.v.}k$ be a tempered distribution?

Q2 Under what conditions on k will $\hat{K} \in L^\infty$? Recall that if $K \in S'$, then $\hat{K} \in S'$ is defined by duality.

Roughly speaking, we we will require k to have decay at infinity and the singularity to have enough cancelation so that a principal value can be defined. Specifically, we define the following size, limit, and cancelation properties.

Definition 2.3. Suppose $k(x) \in L^1_{\text{loc}}(\mathbb{R}^n \backslash \{0\})$.

1. We say k satisfies the size condition if there exists a constant $C_1 > 0$ such that for all $j \in \mathbb{Z}$,

$$\int_{B_{j+1} \backslash B_j} |k(x)| dx \le C_1 < \infty. \tag{2.4}$$

2. We say k satisfies the limit condition if there exists $L \in \mathbb{C}$ such that

$$L = \lim_{j \to -\infty} \int_{B_0 \backslash B_j} k(x) dx. \tag{2.5}$$

3. We say k has the cancelation property if there exists $C_2 > 0$ such that for all $j, R \in \mathbb{Z}$ with $j \le R$,

$$\left| \int_{B_R \backslash B_j} k(x) dx \right| \le C_2. \tag{2.6}$$

Our goal is the following theorem.

Theorem 2.1. *Suppose $k \in L^1_{\text{loc}}(\mathbb{R}^n \backslash \{0\})$, and satisfies Hörmander's condition condition* (2.2)*, the size condition* (2.4)*, the limit condition* (2.5)*, and the cancelation condition* (2.6)*. Then $K = \widehat{\text{p.v.}k}$ gives rise to a singular integral operator T of principal type:*

$$Tf(x) = ((\text{p.v.}k) * f)(x) = \lim_{j \to -\infty} \int_{\mathbb{R}^n \backslash B_j} k(x - y) f(y) dy,$$

defined initially on the Schwartz class S.

To prove this theorem, we will break it up into the following lemmas.

2.1.1. Sufficiency for $K = \text{p.v.}k \in S'$

The following lemma gives sufficient conditions for a locally integrable function k to be a tempered distribution. Roughly speaking, it suffices for k to have some decay at infinity, and any singularity at the origin to be manageable.

Lemma 2.1. *Suppose $k \in L^1_{\text{loc}}(\mathbb{R}^n \backslash \{0\})$ satisfies conditions (2.4) and (2.5). For $\phi \in S$, if we define $K : S \to \mathbb{C}$ as*

$$K(\phi) = (\text{p.v.}k)(\phi) = \lim_{j \to -\infty} \int_{\mathbb{R}^n \backslash B_j} k(x)\phi(x)dx \tag{2.7}$$

then $K = \text{p.v.}k \in S'$.

Proof. We break up $K(\phi)$ into a 'local' and 'global' piece.

$$(\text{p.v.}k)(\phi) = \lim_{j \to -\infty} \int_{B_1 \backslash B_j} k(x)\phi(x)dx + \int_{\mathbb{R}^n \backslash B_1} k(x)\phi(x)dx$$

which we denote by I_1 and I_2, respectively. To bound the 'local piece' I_1, we use the (2.4) and (2.5) as follows:

$$\begin{aligned} I_1 &= \lim_{j \to -\infty} \int_{B_1 \backslash B_j} (k(x)\phi(x) + \phi(0)k(x) - \phi(0)k(x))dx \\ &= \lim_{j \to -\infty} \phi(0) \int_{B_1 \backslash B_j} k(x)dx + \lim_{j \to -\infty} \int_{B_1 \backslash B_j} k(x)(\phi(x) - \phi(0))dx \\ &= \phi(0)L + \lim_{j \to -\infty} \int_{B_1 \backslash B_j} k(x)(\phi(x) - \phi(0))dx. \end{aligned}$$

Since $\phi \in S$, it follows that $\|\nabla\phi\|_\infty$ is bounded. Using this, along with the fact that for $x \in B_1$, we have $|x| \le c_A \rho(x)^{\zeta_-}$, we use (2.4) in the last estimate below:

$$\begin{aligned}
\left| \lim_{j\to-\infty} \int_{B_1\setminus B_j} k(x)(\phi(x)-\phi(0))dx \right| &\le \lim_{j\to-\infty} \int_{B_1\setminus B_j} |k(x)| \|\nabla\phi\|_\infty \cdot |x| dx \\
&\le c_A \|\nabla\phi\|_\infty \lim_{j\to-\infty} \int_{B_1\setminus B_j} |k(x)|\rho(x)^{\zeta_-} dx \\
&= c_A \|\nabla\phi\|_\infty \sum_{j=-\infty}^{0} \int_{B_{j+1}\setminus B_j} |k(x)|\rho(x)^{\zeta_-} dx \\
&\le c_A C_1 \|\nabla\phi\|_\infty \sum_{j=-\infty}^{0} b^{j\zeta_-} = c_A C_1 \left(\frac{1}{1-b^{-\zeta_-}}\right) \|\nabla\phi\|_\infty.
\end{aligned}$$

With $c_A' = c_A \left(\frac{1}{1-b^{-\zeta_-}}\right)$, we have $|I_1| \le |\phi(0)L| + c_A' C_1 \|\nabla\phi\|_\infty < \infty$.

Now we bound I_2. Using (2.4) and the fact that $\phi \in S$,

$$\begin{aligned}
|I_2| = \left| \int_{\mathbb{R}^n\setminus B_1} k(x)\phi(x)dx \right| &\le \int_{\mathbb{R}^n\setminus B_1} |k(x)||\phi(x)| \frac{\rho(x)}{\rho(x)} dx \\
&\le \left(\sup_{x\in\mathbb{R}^n} |\phi(x)|\rho(x) \right) \int_{\mathbb{R}^n\setminus B_1} \frac{|k(x)|}{\rho(x)} dx = \|\phi\|_{0,1} \sum_{j=0}^{\infty} \int_{B_{j+1}\setminus B_j} \frac{|k(x)|}{\rho(x)} dx \\
&\le C_1 \|\phi\|_{0,1} \sum_{j=0}^{\infty} \frac{1}{b^j} = \left(\frac{1}{1-b^{-1}}\right) C_1 \|\phi\|_{0,1}.
\end{aligned}$$

All together, we have:

$$|\text{p.v.}k(\phi)| \le |L\phi(0)| + c_A' C_1 \|\nabla\phi\|_\infty + \left(\frac{1}{1-b^{-1}}\right) C_1 \|\phi\|_{0,1} < \infty.$$

Moreover, $p.v.k$ is easily seen to be a linear mapping $S(\mathbb{R}^n) \to \mathbb{C}$. Lastly, from the this estimate, we also see that if $\phi_j \to \phi$ in S, then $|\text{p.v.}k(\phi_j - \phi)| \to 0$. This means p.v.$k$ is indeed a continuous linear functional on $S(\mathbb{R}^n)$, so p.v.$k \in S'(\mathbb{R}^n)$. $\square$

2.1.2. Sufficiency for $\hat{K} \in L^\infty$

With $K = p.v.k$ established as a tempered distribution, we now seek conditions under which $\widehat{K} \in L^\infty(\mathbb{R}^n)$. To state our main result, we define the truncated kernel as follows. Given a kernel $k \in L^1_{\mathrm{loc}}(\mathbb{R}^n \backslash \{0\})$ and $j, R \in \mathbb{Z}$, with $j < R$, we define the truncated kernel $k_j^R(x) = k(x)\mathbf{1}_{B_R \backslash B_j}(x)$.

If A is a dilation matrix, then its transpose A^* is also a dilation matrix. We will denote ρ_* as the associated quasinorm of A, and the annulus of A^* by $\{B_j^*\}_{j \in \mathbb{Z}}$.

Lemma 2.2. *Suppose $k \in L^1_{\mathrm{loc}}(\mathbb{R}^n \backslash \{0\})$ satisfies Hörmander's condition* (2.2)*, the size condition* (2.4)*, and the cancelation condition* (2.6)*. Then there exists a constant C_k such that for all $j, R \in \mathbb{Z}$ with $j \leq R$ and $\xi \neq 0$,*

$$|\widehat{k_j^R}(\xi)| \leq C_k, \tag{2.8}$$

with C_k independent of j and R. Moreover, $\widehat{\text{p.v.}k} \in L^\infty$.

Now Lemma 2.2 follows from a uniform bound on a specific annulus, given by the following lemma which we will prove later. We define the dilation operator D_A by $D_A k(x) = k(Ax)$. We will need the following fact regarding dilations and the Fourier transform.

$$b^j (D^j_{A^*} \mathcal{F} D^j_A f)(\xi) = \hat{f}(\xi). \tag{2.9}$$

Lemma 2.3. *Define $\beta_A = -\log_b(b-1) - \omega > 0$. Suppose k satisfies the conditions in Lemma 2.2. Then there exists $M \in \mathbb{Z}$ and a constant C, such that for all $\xi \in B^*_{M+1} \backslash B^*_M$,*

$$|\widehat{k_j^R}(\xi)| \leq C,$$

with $j, R \in \mathbb{Z}$ satisfying $R - j \geq \beta_A$.

Proof of Lemma 2.2. Let M be the integer from Lemma 2.3. Suppose for all $\xi \in B^*_{M+1} \backslash B^*_M$, $|\widehat{k^R_j}(\xi)| \leq C$. We now extend this estimate to any $\xi \in \mathbb{R}^n$ using a dilation argument. We first have the relation $D_A(k^R_j)(x) = k^R_j(Ax) = (D_A k)^{R-1}_{j-1}(x)$. Next, we define the scaled dilation operator

$$T_A k(x) = bD_A k(x) = bk(Ax).$$

We will prove that T_A preserves the conditions on k with the same bounds in (2.3), (2.4), and (2.6).

We start with condition (2.3). This requires showing that whenever $\rho(z) \geq \rho(y)b^{2\omega} > 0$,

$$|(T_A k)(z - y) - (T_A k)(z)| = b|k(Az - Ay) - k(Az)| \leq C_L \rho(y)^\gamma \rho(z)^{-(1+\gamma)}.$$

Now since $\rho(z) \geq \rho(y)b^{2\omega}$,

$$\rho(Az) = b\rho(z) \geq b\rho(y)b^{2\omega} = \rho(Ay)b^{2\omega}.$$

Therefore by (2.3),

$$\begin{aligned} b|k(Az - Ay) - k(Az)| &\leq bC_L \rho(Ay)^\gamma \rho(Az)^{-(1+\gamma)} \\ &= C_L b \cdot b^\gamma \rho(y)^\gamma b^{-(1+\gamma)} \rho(z)^{-(1+\gamma)} \\ &= C_L \rho(y)^\gamma \rho(z)^{-(1+\gamma)}. \end{aligned}$$

This shows that (2.3) holds for $D_A k$ with the same constants C_L and γ.

To see that condition (2.4) holds, we let $r \in \mathbb{Z}$. With a change of variables, we obtain:

$$\begin{aligned}\int_{B_{r+1}\setminus B_r} T_A k(x)dx &= \int_{B_{r+1}\setminus B_r} b|k(Ax)|dx = b\int_{B_{r+1}\setminus B_r} |k(Ax)|dx \\ &= b\int_{B_{r+2}\setminus B_{r+1}} |k(u)|b^{-1}du = \int_{B_{r+2}\setminus B_{r+1}} |k(u)|du \leq C_1.\end{aligned}$$

To see condition (2.6) holds, let $r_1, r_2 \in \mathbb{Z}$ with $r_1 < r_2$. Again, by a change of variables, we obtain:

$$\left|\int_{B_{r_2}\setminus B_{r_1}} T_A k(x)dx\right| = \left|\int_{B_{r_2+1}\setminus B_{r_1+1}} k(u)du\right| \leq D_2.$$

With $T_A k$ satisfying the same estimates as k, we can now extend any result for k to $T_A^m k$ (for any $m \in \mathbb{Z}$), as long as these results depend only on the constants appearing in the conditions (2.3), (2.4), and (2.6), that is, C_L, γ, C_1, and C_2. Then we conclude that for all $m \in \mathbb{Z}$ and $\xi \in B^*_{M+1}\setminus B^*_M$,

$$|\widehat{(T_A^m k)_j^R}(\xi)| \leq C, \tag{2.10}$$

with the same constant $C = C(A, C_1, C_2, C_3, M)$. Therefore for any $j \in \mathbb{Z}$, the dilated kernel $T_A^m k$ satisfies the same estimates as k, and the estimates for $\widehat{k_j^R}(\xi)$ also hold for $\widehat{T_A^m(k_j^R)}(\xi)$. This observation allows us to extend our bounds above to all $\xi \in \mathbb{R}^n$ as follows.

Let $z \neq 0$. Then there exists $q \in \mathbb{Z}$ such that $z \in B^*_{M+1-q} \backslash B^*_{M-q}$, which is equivalent to $(A^*)^q z \in B^*_M \backslash B^*_{M-1}$. Then by (2.9), we have the following identity:

$$
\begin{aligned}
|\widehat{k_j^R}(z)| &= |\mathcal{F}(k_j^R)(z)| = |b^q (D^q_{A^*} \mathcal{F} D^q_A) k_j^R(z)| \\
&= b^q |(D^q_A(k_j^R))^\wedge(A^{*q} z)| = |(T^q_A k_j^R)^\wedge(A^{*q} z)| \\
&= |((T^q_A k)^{R-q}_{j-q})^\wedge((A^*)^q z)| \leq C.
\end{aligned}
$$

The last inequality holds since $T^q_A k$ is a kernel satisfying the same conditions as k with the same constants. Therefore we can use (2.10) with $\xi = (A^*)^q z \in B^*_{M+1} \backslash B^*_M$. With j, R arbitrary, this uniform bound holds for all truncations. □

It now remains to prove Lemma 2.3. This requires two propositions.

Proposition 2.2. *Suppose* $\beta_A = -\log_b(b-1) - \omega > 0$. *If* $k \in L^1_{\text{loc}}(\mathbb{R}^n \backslash \{0\})$ *satisfies the size condition* (2.4), *then there exists* $C_k > 0$ *such that for all* $j, R \in \mathbb{Z}$ *with* $0 < R - j \leq \beta_A$, $\|\widehat{k_j^R}\|_\infty \leq C_K$.

Proof. By assumption, $\beta_A > 0$. (This means b is quite close to 1.) Then by (2.4),

$$
\begin{aligned}
\widehat{k_j^R}(\xi)| &= \left| \int k_j^R(x) e^{-2\pi i \langle x, \xi \rangle} dx \right| \leq \int_{B_R \backslash B_j} |k(x)| dx \\
&= \sum_{m=j}^{R} \int_{B_{m+1} \backslash B_m} |k(x)| dx \leq C_1(R-j) \leq C_1 \beta_A = C_k.
\end{aligned}
$$

□

We now provide an estimate on a Hörmander type integral that we will need when we look at the case when $R - j > \beta_A$. If $\beta_A < 0$, then the above lemma is not needed at all.

Proposition 2.3. *Let $j, R \in \mathbb{Z}$ satisfy $R - j \geq \beta_A > 0$. Suppose $k \in L^1_{\text{loc}}(\mathbb{R}^n \backslash \{0\})$ satisfies Hörmander's condition* (2.2) *and the size condition* (2.4)*. If $r \in \mathbb{Z}$ satisfies $B_{r+2\omega} \subseteq B_j$, then there exists C_3, depending only on C_1, C_H, A such that for all $y \in B_r$,*

$$\int_{\mathbb{R}^n} |k_j^R(x-y) - k_j^R(x)| dx \leq C_3. \tag{2.11}$$

Proof. If $R - j \geq \beta_A$, then we have the inequality

$$b^j \leq b^{R+\omega}(b-1).$$

We denote the integral in (2.11) by I and we break I into the following five pieces.

$$\begin{aligned}
I_1 &= \int_{U_1} |k_j^R(x-y) - k_j^R(x)| dx \qquad \text{where } U_1 = (B_R + y) \backslash B_R, \\
I_2 &= \int_{U_2} |k_j^R(x-y) - k_j^R(x)| dx \qquad \text{where } U_2 = B_R \backslash (B_R + y), \\
I_3 &= \int_{U_3} |k_j^R(x-y) - k_j^R(x)| dx \qquad \text{where } U_3 = (B_j + y) \backslash B_j, \\
I_4 &= \int_{U_4} |k_j^R(x-y) - k_j^R(x)| dx \qquad \text{where } U_4 = B_j \backslash (B_j + y), \\
I_5 &= \int_{U_5} |k_j^R(x-y) - k_j^R(x)| dx \qquad \text{where } U_5 = B_R \cup (B_R + y)) \backslash \left((\bigcup_{n=1}^{4} U_n) \cup B_j \cup (B_j + y) \right)
\end{aligned}$$

Estimate on I_1. If $y \in B_r$, then since $r < j < R$, we have the the containment

$$y + B_R \subset B_r + B_R \subset B_{R+\omega}.$$

Since the support of k_j^R is contained in B_R, we have

$$\begin{aligned} I_1 &= \int_{U_1} |k_j^R(x-y) - k_j^R(x)| dx = \int_{U_1} |k_j^R(x-y)| dx \\ &\leq \int_{B_{R+\omega} \backslash B_R} |k_j^R(x-y)| dx = \sum_{m=0}^{\omega-1} \int_{B_{R+m+1} \backslash B_{R+m}} |k_j^R(x-y)| dx. \end{aligned}$$

Fix $m \in \{0, 1, \ldots, \omega - 1\}$. If $x \in B_{R+m+1} \backslash B_{R+m}$, then $\rho(x) \leq b^{R+m}$, giving

$$\rho(x-y) \leq b^{\omega}(\rho(x) + \rho(y)) \leq b^{\omega}(b^{R+m} + b^{r-1}).$$

Next, we provide a lower bound using (1.10),

$$\rho(x-y) \geq b^{-\omega}\rho(x) - \rho(y) = b^{-\omega} b^{R+m} - \rho(y) \geq b^{-\omega} b^{R+m} - b^{r-1}.$$

Therefore if $x \in B_{R+m+1} \backslash B_{R+m}$,

$$b^{-\omega} b^{R+m} - b^{r-1} \leq \rho(x-y) \leq b^{\omega}(b^{R+m} - b^{r-1}).$$

After a change of variables, we have:

$$\int_{B_{R+m+1} \backslash B_{R+m}} |k_j^R(x-y)| dx \leq \int_{b^{-\omega} b^{R+m} - b^{r-1} \leq \rho(z) \leq b^{\omega}(b^{R+m} - b^{r-1})} |k_j^R(z)| dz.$$

Since we have a truncated kernel, we can simplify the domain by just looking at $z \in B_R$. Furthermore, by the definition of ω and $m \leq \omega$, we have $b^{R-\omega} \leq b^{R+m+1-\omega} - b^r$, which allows us to restrict our domain to $B_R \backslash B_{R-\omega+1}$. Finishing with (2.4),

$$\begin{aligned}
\int_{B_{R+m+1} \backslash B_{R+m}} |k_j^R(x-y)|dx &\leq \int_{b^{R-\omega} \leq \rho(z) \leq b^R} |k(z)|dz = \int_{B_R \backslash B_{R-\omega+1}} |k(z)|dz \\
&= \sum_{i=0}^{\omega-2} \int_{B_{R-\omega+1+i+1} \backslash B_{R-\omega+1+m}} |k(z)|dz \\
&\leq (\omega-1)C_1.
\end{aligned}$$

Since this estimate holds for each m, we have:

$$I_1 \leq \sum_{m=0}^{\omega-1} \int_{B_{R+m+1} \backslash B_{R+m}} |k_j^R(x-y)|dx \leq \sum_{m=0}^{\omega-1} (\omega-1)C_1 = C_1(\omega-1)\omega.$$

Estimate on I_2. Since $U_2 = B_R \backslash (B_R + y)$, and $\text{supp}(k_j^R(x-y)) \subseteq B_R + y$,

$$I_2 = \int_{U_2} |k_j^R(x-y) - k_j^R(x)|dx = \int_{B_R \backslash (B_R+y)} |k_j^R(x)|dx.$$

Now for $y \in B_r$, $B_{R-\omega} - y \subseteq B_{R-\omega} - B_y \subseteq B_{R-\omega+\omega} = B_R$. (Since $r \leq j - 2\omega < j < R$, so $R - \omega > r$, the last inclusion holds.) Therefore:

$$B_{R-\omega} - y \subseteq B_R \Rightarrow B_{R-\omega} \subseteq B_R + y \Rightarrow B_R \backslash (B_R + y) \subseteq B_R \backslash B_{R-\omega}.$$

We now complete the estimate on I_2 using (2.4).

$$\begin{aligned}
I_2 &= \int_{B_R \backslash (B_R+y)} |k_j^R(x)|dx \leq \int_{B_R \backslash B_{R-\omega}} |k(x)|dx \leq (\omega-1)C_1 \\
&= \sum_{m=0}^{\omega-1} \int_{B_{R-\omega+m+1} \backslash B_{R-\omega+m}} |k(x)|dx \leq \omega C_1.
\end{aligned}$$

Estimate on I_3. Since $U_3 = (B_j + y)\backslash B_j$, and $k_j^R(x-y) = 0$ on $B_j + y$, we have:

$$I_3 = \int_{(B_j+y)\backslash B_j} |k_j^R(x)|dx.$$

Since $r < j$, for $y \in B_r$, we have $B_j + y \subseteq B_j + B_r \subseteq B_{j+\omega}$. Therefore $(B_j + y)\backslash B_j \subseteq B_{j+\omega}\backslash B_j$. Again by (2.4),

$$\begin{aligned} I_3 &\leq \int_{(B_j+y)\backslash B_j} |k(x)|dx \leq \int_{B_{j+\omega}\backslash B_j} |k(x)|dx \\ &= \sum_{m=0}^{\omega-1} \int_{B_{j+m+1}\backslash B_{j+m}} |k(x)|dx \leq \omega C_1. \end{aligned}$$

Estimate on I_4. Since $U_4 = B_j\backslash(B_j + y)$, and $k_j^R(x) = 0$ on B_j,

$$I_4 = \int_{B_j\backslash(B_j+y)} |k_j^R(x-y)|dx.$$

Since $r+2\omega \leq j$, if $y \in B_r$, we have $B_{j-\omega} - y \subseteq B_{j-\omega} - B_r \subseteq B_{j-\omega+\omega} = B_j$. Therefore $B_{j-\omega} \subseteq B_j + y$ for all $y \in B_r$, thus giving $B_j\backslash(B_j + y) \subseteq B_j\backslash B_{j-\omega}$. Another fact is this:

$$x \in B_j\backslash B_{j-\omega} \text{ and } y \in B_r \Rightarrow x - y \in B_{j+\omega}.$$

Since k_j^R is 0 on B_j, a change of variables then gives

$$\begin{aligned} I_4 &= \int_{B_j\backslash(B_j+y)} |k_j^R(x-y)|dx \leq \int_{B_j\backslash B_{j-\omega}} |k_j^R(x-y)|dx \\ &\leq \int_{b^j \leq \rho(z) \leq b^{j+\omega}} |k_j^R(z)|dz \leq \int_{B_{j+\omega}\backslash B_j} |k(z)|dz \\ &= \sum_{m=0}^{\omega-1} \int_{B_{j+m+1}\backslash B_{j+m}} |k(z)|dz \leq C_1\omega. \end{aligned}$$

Estimate on I_5. Since $U_5 \subset (B_R \cup (B_R + y)) \backslash (U_1 \cup U_2 \cup B_j \cup (B_j + y))$, we have:

$$I_5 = \int_{U_5} |k_j^R(x-y) - k_j^R(x)| dx \leq \int_{B_R \backslash B_j} |k_j^R(x-y) - k_j^R(x)| dx.$$

Observe that $B_R \backslash B_j \subseteq B_j^c$. Next, for $y \in B_r$, we see that:

$$E_y = \{z : \rho(z) < b^{2\omega} \rho(y)\} \subseteq \{z : \rho(z) \leq b^{2\omega + r}\} = B_{r+2\omega}.$$

Therefore $B_{r+2\omega}^c \subseteq E_y^c$. This gives the following containment:

$$E_y \subseteq B_{r+2\omega} \subseteq B_j \qquad \Rightarrow \qquad E_y^c \supseteq B_{r+2\omega}^c \supseteq B_j^c.$$

Using condition (2.2):

$$\begin{aligned} I_5 &= \int_{B_R \backslash B_j} |k_j^R(x-y) - k_j^R(x)| dx \leq \int_{B_j^c} |k_j^R(x-y) - k_j^R(x)| dx \\ &\leq \int_{E_y^c} |k_j^R(x-y) - k_j^R(x)| dx \leq C_H. \end{aligned}$$

In summary, given $j, R, r \in \mathbb{Z}$ as defined above, we have shown that for any $y \in B_r$, the following estimate holds:

$$\int_{\mathbb{R}^n} |k_j^R(x-y) - k_j^R(x)| dx \leq C_1 \omega(\omega - 1) + 3C_1\omega + C_H = C_3.$$

$\square$

Armed with this Hörmander-type estimate, we can now prove Lemma 2.3.

Proof of Lemma 2.3. By Proposition 2.2, we only need to consider the case when $R - j \geq \beta_A$. We will now show that there exists $M \in \mathbb{Z}$ and a constnat C such that for all R and j such that $R - j \geq \beta_A$ and $\xi \in B^*_{M+1} \backslash B^*_M$, we have:

$$|\widehat{k_j^R}(\xi)| \leq C,$$

where C depends only on the matrix A and the constants C_1, C_2, C_3, M.

Without loss of generality, we can take $c_A > \sqrt{2}$ in the inequality (1.11). We define $\delta = \log_b(c_A^2/2)$ and

$$M = \inf \left\{ n \in \mathbb{Z} : n \geq \frac{1}{\zeta_-}(\delta + 2\omega\zeta_+ \right\}. \tag{2.12}$$

We first note that $M \geq 1$, since $\delta > 0$. Now fix $\xi \in B^*_{M+1} \backslash B^*_M$. We have three cases to consider for j, R: when $j < R \leq 0$, $j < 0 < R$, and $0 \leq j < R$.

Case 1. Assume $j < 0 < R$. Then $|\widehat{k_j^R}(\xi)| \leq |\widehat{k_j^0}(\xi)| + |\widehat{k_0^R}(\xi)|$, and we estimate each piece separately. Given $\xi \in B^*_{M+1} \backslash B^*_M$, we define $y = -\frac{\xi}{2|\xi|^2}$. We first claim $y \in B_{-2\omega}$. Indeed, the definition of M implies the following inequality:

$$\frac{c_A}{2} b^{2\omega\zeta_+} \leq \frac{1}{c_A} b^{M\zeta_-}.$$

Then since $\xi \in B^*_{M+1} \backslash B^*_M$, and $M \geq 0$, by (1.11),

$$\frac{c_A}{2} b^{2\omega\zeta_+} \leq \frac{1}{c_A} b^{M\zeta_-} = \frac{\rho(\xi)^{\zeta_-}}{c_A} \leq |\xi|.$$

Next, by taking the reciprocal of the inequality above. we obtain:

$$|y| = \frac{1}{2|\xi|} \leq \frac{b^{-2\omega\zeta_+}}{c_A}.$$

Now if $\rho(y) \leq 1$, (we can assume this since M is large enough) we have the inequality:

$$\frac{1}{c_A}\rho(y)^{\zeta_+} \leq |y| \leq \frac{b^{-2\omega\zeta_+}}{c_A}.$$

The inequality above implies $\rho(y) \leq b^{-2\omega}$, which means $y \in B_{-2\omega}$.

Next, we bound $|\widehat{k_0^R}(\xi)|$. For ξ, y as above, $\langle \xi, y\rangle = -\frac{1}{2}$. Then because our choice of M forces $y \in B_{-2\omega}$, by the claim above, we can use (2.11) to obtain

$$\begin{aligned}
|\widehat{k_0^R}(\xi)| &= \frac{1}{2}|\widehat{k_0^R}(\xi)(1 - e^{-2\pi i\langle y,\xi\rangle})| \\
&= \frac{1}{2}\left|\int_{\mathbb{R}^n}(k_0^R(x) - k_0^R(x-y))e^{-2\pi i\langle \xi,x\rangle}dx\right| \\
&\leq \frac{1}{2}\int_{\mathbb{R}^n}|k_0^R(x) - k_0^R(x-y)|dx \leq \frac{1}{2}C_3.
\end{aligned}$$

We now bound $\widehat{k_j^0}(\xi)$. We have

$$\begin{aligned}
|\widehat{k_j^0}(\xi)| &= \left|\int k_j^0(x)(e^{-2\pi i\langle x,\xi\rangle} - 1) + k_j^0(x)dx\right| \\
&\leq \int |k_j^0(x)||e^{-2\pi i\langle x,\xi\rangle} - 1|dx + \left|\int k_j^0(x)dx\right| \\
&\leq C\int_{B_0\setminus B_j}|k(x)|\cdot|x|\cdot|\xi|dx + C_2 \\
&\leq C_A|\xi|\sum_{m=0}^{-j-1}\left(\int_{B_{j+m+1}\setminus B_{j+m}}|k(x)|\rho(x)^{\zeta_-}dx\right) + C_2 \\
&= C_AC_1|\xi|b^{j\zeta_-}\sum_{m=0}^{-j-1}b^{m\zeta_-} + C_2 = C_AC_1|\xi|b^{j\zeta_-}\left(\frac{b^{-j\zeta_-}-1}{b^{\zeta_-}-1}\right) + C_2 \\
&= \frac{c_AC_1}{b^{\zeta_-}-1}\rho(\xi)^{\zeta_+}(1 - b^{j\zeta_-}) + C_2 = \frac{c_AC_1}{b^{\zeta_-}-1}b^{M\zeta_+}(1 - b^{j\zeta_-}) + C_2.
\end{aligned}$$

This completes this case, with $|\widehat{k_j^R}(\xi)| \leq C$, with $C = C(A, C_1, C_2, C_3, M)$.

Case 2. Assume $j < R \leq 0$. Then by the same Lipshitz argument in case 1, we have:

$$
\begin{aligned}
|\widehat{k_j^R}(\xi)| &\leq C \int_{B_R \setminus B_j} |k(x)| \cdot |x| \cdot |\xi| dx + C_2 \\
&\leq C \int_{B_o \setminus B_j} |k(x)| \cdot |x| \cdot |\xi| dx + C_2 \\
&\leq C(A, C_1, C_2, M).
\end{aligned}
$$

Case 3. Assume $0 \leq j < R$. Then if $\xi \in B_M^* \setminus B_{M-1}^*$, we have $y = -\frac{\xi}{2|\xi|^2} \in B_{-2\omega} \subseteq B_{j-2\omega}$, since $j \geq 0$. Therefore we can apply (2.11) and the argument in case 1 to estimate $\widehat{k_0^R}(\xi)$, obtaining

$$
\begin{aligned}
|\widehat{k_j^R}(\xi)| &= \frac{1}{2} |\widehat{k_j^R}(\xi)(1 - e^{-2\pi i \langle y, \xi \rangle})| \\
&= \frac{1}{2} \left| \int (k_j^R(x) - k_j^R(x - y)) e^{-2\pi i \langle \xi, x \rangle} dx \right| \\
&\leq \frac{1}{2} \int |k_j^R(x) - k_j^R(x - y)| dx \leq \frac{1}{2} C_3.
\end{aligned}
$$

In summary, given $j, R \in \mathbb{Z}$ and $\xi \in B_{M+1}^* \setminus B_M^*$, we have shown the existence of $C = C(A, C_1, C_2, C_3, M)$ such that $|\widehat{\text{p.v.} k_j^R}(\xi)| \leq C$. $\square$

2.2. Molecules and the Molecular Norm

We now define anisotropic molecule, which will be seen as the image of an atom under a singular integral operator. Molecules also give another decomposition of Hardy spaces.

Fix A. We recall that a triple (p, q, s) is admissible if $p \in (0, 1]$, $q \in [1, \infty]$ with $p < q$, and $s \in \mathbb{N}$ such that $s \geq \lfloor (\frac{1}{p} - 1) \frac{\log b}{\log \lambda_-} \rfloor$.

2.2.1. Definition of a Molecule

Definition 2.4. Let (p, q, s) be admissible and let d satisfy

$$d > s \frac{\log \lambda_+}{\log b} + 1 - \frac{1}{q}. \tag{2.13}$$

Define $\theta = \dfrac{\frac{1}{p} - \frac{1}{q}}{d}$. Then $M \in L^q(\mathbb{R}^n)$ is a (p, q, d) molecule centered at $x_0 \in \mathbb{R}^n$ if:

(i) $N(M) = \|M\|_q^{1-\theta} \|\rho(x - x_0)^d M(x)\|_q^\theta < \infty$.

(ii) $\displaystyle\int_{\mathbb{R}^n} x^\beta M(x) dx = 0$ for all β such that $|\beta| \leq s$.

We call $N(M)$ the molecular norm of M.

Remark 2.2. 1. Whenever we call M a (p, q, d) molecule, we are implicitly assuming we have already fixed an admissible triplet (p, q, s) associated with a dilation A as well as d satisfying (2.13).

2. The integral in (ii) is absolutely convergent simply by assuming (i). Indeed, since $M \in L^q(\mathbb{R}^n)$, then on a compact set K, we also have $\chi_K M \in L^1(\mathbb{R}^n)$. So

it suffices to show the integral is finite away from the origin:

$$\int_{B_1^c} |x|^s |M(x)| dx < \infty.$$

Now this integral can be estimated using Holder's inequality, with $\frac{1}{q} + \frac{1}{q'} = 1$:

$$\begin{aligned}\int_{B_1^c} |x|^s |M(x)| dx &\leq C_A^s \int_{B_1^c} \rho(x)^{s\zeta_+} |M(x)| dx = \int_{B_1^c} |\rho(x)^d M(x)| \rho(x)^{s\zeta_+ - d} dx \\ &\leq C_A^s \left(\int_{B_1^c} \rho(x)^{dq} |M(x)|^q dx \right)^{1/q} \left(\int_{B_1^c} \rho(x)^{q'(s\zeta_+ - d)} dx \right)^{1/q'} \\ &\leq \|\rho(x)^d M(x)\|_q \left(\int_{B_1^c} \rho(x)^{q'(s\zeta_+ - d)} dx \right)^{1/q'} .\end{aligned}$$

Assuming (i) to be true, we only have to show the last integral above is finite. This is true if and only if the exponent is small enough: $q'(s\zeta_+ - d) < -1$. This also explains the necessity of condition (2.13) on d. Lastly, we observe that the above quantities remain finite if we shift $\rho(x)$ to $\rho(x - x_0)$, so the above analysis can readily be extended to the case when $x_0 \neq 0$.

3. Let M be a (p, q, d) molecule. If for some $\overline{d}$,

$$s \frac{\log \lambda_+}{\log b} + 1 - \frac{1}{q} < \overline{d} < d,$$

then M is also a $(p, q, \overline{d})$ molecule. Without loss of generality, let $x_0 = 0$. We only have to check condition (i) in the definition of the molecule, with $\overline{\theta} = \frac{\frac{1}{p} - \frac{1}{q}}{\overline{d}}$. This is satisfied if the following term is finite:

$$\|\rho(x - x_0)^{\overline{d}} M(x)\|_q^{\overline{\theta}} < \infty.$$

Again, we only need to show that the above term is finite over B_1^c. But on B_1^c, we have the following inequality:

$$\rho(x)^{\overline{d}}M(x) \leq \rho(x)^d M(x).$$

Therefore the above q-norm is also finite, and (i) is satisfied.

The next result shows that every atom is also a molecule whose molecular norm is uniformly bounded as well.

Lemma 2.4. *Let (p,q,s) be admissible. Then every (p,q,s) atom $a(x)$ supported on an annulus $x_0 + B_k$ is also a (p,q,d) molecule centered at x_0. Furthermore, there exists a constant $C = C(A,d)$ such that for all atoms a,*

$$N(a) \leq C.$$

Proof. Let a be a (p,q,s) atom supported on an annulus $x_0 + B_k$. Condition (ii) holds immediately, so we only need to show (i). The first term in (i) is estimated by the size condition on the atom:

$$\|a\|_q^{1-\theta} \leq |B_k|^{(1-\theta)(\frac{1}{q}-\frac{1}{p})}.$$

The second term in (i) is estimated as follows:

$$\begin{aligned}
\|\rho(x-x_o)^d a(x)\|_q &= \left(\int_{x_o+B_k} |\rho(x-x_0)^{dq} a(x)^q| dx\right)^{1/q} \\
&\leq (b^{k-1})^d \left(\int_{x_o+B_k} |a(x)|^q dx\right)^{1/q} \\
&= b^{-d}|B_k|^d \|a\|_q \leq |B_k|^d |B_k|^{\frac{1}{q}-\frac{1}{p}}.
\end{aligned}$$

All together,

$$N(a) = \|a\|_q^{1-\theta} \|\rho(x-x_0)^d a(x)\|_q^\theta \leq b^{-d} |B_k|^{(1-\theta)(\frac{1}{q}-\frac{1}{p})} |B_k|^{(d+\frac{1}{q}-\frac{1}{p})\theta} = b^{-d} |B_k|^{\frac{1}{q}-\frac{1}{p}+d\theta} = b^{-d}.$$

Note that the power of the $|B_k|$ term is precisely 0 due to the definition of θ. $\square$

2.2.2. Molecular Decomposition of H_A^p

We now prove that the H^p-norms of molecules are bounded by their molecular norms. To see this, we will provide an atomic decomposition into (p, q, s) atoms and show the resulting coefficients are uniformly bounded in ℓ^p. Our goal is the following result.

Theorem 2.2. *(Molecular Decomposition of Hardy Spaces) Let (p, q, s) be a admissible triplet and let d satisfy* (2.13). *Then $f \in H^p(\mathbb{R}^n)$ if and only if there exist (p, q, d)-molecules $\{M_j\}$ such that the following converges in S' and in H^p-norm:*

$$f = \sum_j M_j \qquad \textit{and} \qquad \sum_j N(M_j)^p < \infty.$$

Furthermore there exist constants C_1, C_2, independent of f, such that given the above series, we have:

$$\|f\|_{H^p}^p \leq C_1 \sum_j N(M_j)^p \qquad \textit{and} \qquad \sum_j N(M_j)^p \leq C_2 \|f\|_{H^p}^p.$$

This theorem follows directly from $\|M\|_{H^p} \leq CN(M)$, which is Theorem 2.3. A weaker version of this appears in [Bow03] in which a more specialized definition of molecules is used, though our result follows from a direct iteration of that result. We need two preliminary results from Chapter 1.9 of [Bow03] involving projections.

Definition 2.5. Let $s \in \mathbb{N}$ and $\mathcal{B} = \{x + B_j : x \in \mathbb{R}^n, j \in \mathbb{Z}\}$. Define P_s to be the space of polynomials, in n variables, of degree at most s. If $B \in \mathcal{B}$, we define π_B as the natural projection defined by the Riesz Lemma:

$$\int_B (\pi_B f(x)) Q(x) dx = \int_B f(x) Q(x) dx, \qquad \text{for all } f \in L^1(B) \text{ and } Q \in P_s.$$

We also define $\|f\|_{L^1(B)} = \int_B |f(x)| dx$.

To obtain the properties of π_B for a general ellipsoid B, we start with π_{B_0} on the unit ellipsoid B_0, and extend its properties by dilation and translations.

Proposition 2.4. *Let $Q = \{Q_\alpha\}_{|\alpha| \le s}$ be an orthonormal basis of P_s in $L^2(B_0)$-norm:*

$$\langle Q_\alpha, Q_\beta \rangle = \int_{B_0} Q_\alpha(x) \overline{Q_\beta(x)} dx = \delta_{\alpha,\beta}.$$

1. *$\pi_{B_0} : L^1(B_0) \to P_s$ is given by*

$$\pi_{B_0} f = \sum_{|\alpha| \le s} \left(\int_{B_0} f(x) \overline{Q_\alpha(x)} dx \right) Q_\alpha. \tag{2.14}$$

2. *If $j \in \mathbb{Z}$, then $\pi_{B_j} : L^1(B_j) \to P_s$ is given by*

$$\pi_{B_j} f = \left(D_A^{-j} \pi_{B_0} D_A^j \right) f. \tag{2.15}$$

   If $B = y + B_j$, then $\pi_B : L^1(B) \to P_s$ is given by

$$\pi_B f = \left(T_y \pi_{B_j} T_{-y} \right) f. \tag{2.16}$$

3. There exists C_0, depending only on s and Q, such that for all $B \in \mathcal{B}$, given $x \in B$,

$$|\pi_B f(x)| \leq C_0 \int_B |f| \frac{dx}{|B|}. \tag{2.17}$$

4. Let $\tilde{\pi}_B = \mathrm{Id} - \pi_B$ be the complementary projection. Then for all $B \in \mathcal{B}$, $\tilde{\pi}_B : L^q(B) \to L^q(B)$ is bounded, with

$$\|\tilde{\pi}_B(f)\|_{L^q(B)} \leq (1 + C_0)\|f\|_{L^q(B)}. \tag{2.18}$$

Furthermore, for all α with $|\alpha| \leq s$,

$$\int_B x^\alpha \cdot (\tilde{\pi}_B f)(x) dx = 0.$$

Lemma 2.5. *Let M be a (p, q, d) molecule centered at x_0.*

1. Then $\|\pi_j M\|_{L^1(B_j)} \to 0$ as $j \to \infty$.

2. Define $g_j = (\tilde{\pi}_{B_j} M)\boldsymbol{1}_{B_j} = (M - \pi_{B_j} M)\boldsymbol{1}_{B_j}$. Then $g_j \to M$ in L^1 as $j \to \infty$.

We now show that every (p, q, d) molecule is in $H^p(\mathbb{R}^n)$, and that its H^p-norm is dominated by its molecular norm.

Theorem 2.3. *Let (p, q, s) be a admissible triple and let d satisfy* (2.13)*. Then there is a constnat C such that for all (p, q, d) molecules M, we have $\|M\|_{H^p} \leq CN(M)$.*

Proof. Let M be a (p, q, d) molecule. Without loss of generality, we assume $N(M) = \|M\|_q^{1-\theta} \|M(u)\rho(u)^d\|_q^\theta = 1$. Define σ by

$$\|M\|_q = \sigma^{\frac{1}{q} - \frac{1}{p}},$$

and choose $k \in \mathbb{Z}$ such that $b^k \leq \sigma < b^{k+1}$. From the previous lemmas, we have the following expression for M, with convergence in L^1:

$$M = g_k + \sum_{j=k}^{\infty}(g_{j+1} - g_j).$$

Note that for each j, g_j has vanishing moments of order up to s, and has compact support. We will decompose M as follows:

$$g_k = \mu_k a_k \text{ and } \qquad g_{j+1} - g_j = \mu_j a_j,$$

where $(\mu_j)_{j=k}^{\infty} \in l^p$ has a uniform norm independent of M and $(a_j)_{j=k}^{\infty}$ is a sequence of (p, q, s) atoms.

We start with $g_k = (M - \pi_k M)\mathbf{1}_{B_k}$. With C_0 as in Proposition 2.4,

$$\|g_k\|_{L^q(B_k)} \leq \|M\|_{L^q(B_k)} + \|\pi_k M\|_{L^q(B_k)} \leq (1 + C_0)\|M\|_{L^q(B_k)}.$$

Scaling the measure, we obtain

$$\|g_k\|_{L^q\left(\frac{\chi_{B_k}}{|B_k|}dx\right)} = \left(\int_{B_k} |g_k(x)|^q \frac{dx}{|B_k|}\right)^{1/q} \leq (1 + C_0)\|M\|_{L^q(\frac{dx}{|B_k|})}.$$

Note that because $\frac{1}{q} - \frac{1}{p} < 0$, we have $\sigma \geq b^k \Rightarrow \sigma^{\frac{1}{q} - \frac{1}{p}} \leq b^{k(\frac{1}{q} - \frac{1}{p})}$. Continuing our estimate using the definition of σ, we have

$$\begin{aligned}\|M\|_{L^q(\frac{dx}{|B_k|})} &= |B_k|^{-1/q}\|M\|_q = |B_k|^{-1/q}\sigma^{\frac{1}{q} - \frac{1}{p}} \\ &\leq |B_k|^{-\frac{1}{q}}|B_k|^{\frac{1}{q} - \frac{1}{p}} = |B_k|^{\frac{1}{p}}.\end{aligned}$$

Therefore we have:

$$\|g_k\|_q \le (1+C_0)|B_k|^{\frac{1}{q}-\frac{1}{p}},$$

which gives $g_k = \mu_k a_k$ where a_k is a (p,q,s) atom and $\mu_k = 1 + C_0$. For $j > k$, we have

$$g_{j+1} - g_j = \mathbf{1}_{B_{j+1}\setminus B_j} M - \pi_{B_{j+1}} M + \pi_{B_j} M.$$

We start with the first term.

$$\begin{aligned}
\|M\mathbf{1}_{B_{j+1}\setminus B_j}\|_{L^q(\frac{dx}{|B_{j+1}|})} &= |B_{j+1}|^{-1/q}\left(\int_{B_{j+1}\setminus B_j} |M(x)|^q \left(\frac{\rho(x)}{b^j}\right)^{dq} dx\right)^{1/q} \\
&= |B_{j+1}|^{-\frac{1}{q}} b^{-jd}\left(\int_{B_{j+1}\setminus B_j} |M(x)|^q \rho(x)^{dq} dx\right)^{1/q} \\
&\le b^d |B_{j+1}|^{-\frac{1}{q}} |B_{j+1}|^{-d} \|M(x)\rho(x)^d\|_q = b^d |B_{j+1}|^{-\frac{1}{q}-d} \|M\|_q^{\frac{\theta-1}{\theta}} \\
&= b^d |B_{j+1}|^{-\frac{1}{q}-d} \sigma^{(\frac{1}{q}-\frac{1}{p})(\frac{\theta-1}{\theta})} = b^d |B_{j+1}|^{-\frac{1}{q}-d} \sigma^{d(1-\theta)} \\
&\le b^d |B_{j+1}|^{-\frac{1}{q}-d} b^{(k+1)d(1-\theta)} = b^d |B_{j+1}|^{-\frac{1}{p}} b^{(j+1)(\frac{1}{p}-\frac{1}{q}-d)} b^{(k+1)d(1-\theta)} \\
&= b^d |B_{j+1}|^{-\frac{1}{p}} b^{(k-j)d(1-\theta)}.
\end{aligned}$$

Setting $a = d(1-\theta) > 0$, we obtain

$$\|M\mathbf{1}_{B_{j+1}\setminus B_j}\|_q \le b^d |B_{j+1}|^{\frac{1}{q}-\frac{1}{p}} b^{(j-k)(-a)}.$$

Next, we estimate $\pi_{B_j} M$. It is given by

$$\pi_{B_j} M(x) = \sum_{|\alpha|\le s}\left(\int_{B_j} M(u)\overline{Q_\alpha(A^{-j}u)}\frac{du}{b^j}\right) Q_\alpha(A^{-j}x).$$

By Minkowski's inequality,

$$\|\pi_{B_j} M\|_{L^q(B_j)} \leq \sum_{|\alpha|\leq s} b^{-j} \left| \int_{B_j} M(u)\overline{Q_\alpha(A^{-j}u)}du \right| \|D_A^{-j} Q_\alpha\|_{L^q(B_j)}.$$

Let $C(Q)$ a uniform bound for $\|Q_\alpha\|_{L^q(B_0)}$. By a change of variables, we have $\|D_A^{-j} Q_\alpha\|_{L^q(B_j)} = b^{\frac{j}{q}} \|Q_\alpha\|_{L^q(B_0)} \leq C(Q) b^{\frac{j}{q}}$. Next, since M has vanishing moments, and $\frac{1}{q} + \frac{1}{q'} = 1$,

$$\begin{aligned}
\left| \int_{B_j} M(u)\overline{Q_\alpha(A^{-j}u)}du \right| &= \left| \int_{B_j^c} M(u)\overline{Q_\alpha(A^{-j}u)}du \right| \leq \int_{B_j^c} |M(u)||Q_\alpha(A^{-j}u)|du \\
&\leq C(Q) \int_{B_j^c} |M(u)||A^{-j}u|^s du \leq C(Q) c_A \int_{B_j^c} |M(u)| \rho(A^{-j}u)^{s\zeta_+} du \\
&= C(Q) c_A b^{-js\zeta_+} \int_{B_j^c} |M(u)| \rho(u)^{s\zeta_+} du \\
&= C(Q) c_A b^{-js\zeta_+} \int_{B_j^c} |M(u)| \rho(u)^d \rho(u)^{s\zeta_+ - d} du \\
&\leq C(Q) c_A b^{-js\zeta_+} \left(\int_{B_j^c} |M(u)|^q \rho(u)^{dq} du \right)^{1/q} \left(\int_{B_j^c} \rho(u)^{q'(s\zeta_+ - d)} du \right)^{1/q'}.
\end{aligned}$$

The first integral in the last expression can be computed as follows:

$$\begin{aligned}
\left(\int_{B_j^c} |M(u)|^q \rho(u)^{dq} du \right)^{1/q} &\leq \|M(x)\rho(x)^d\|_q = \|M\|_q^{\frac{\theta-1}{\theta}} \\
&= \sigma^{\left(\frac{1}{q}-\frac{1}{p}\right)\left(\frac{\theta-1}{\theta}\right)} = \sigma^{d(1-\theta)} \leq b^{kd(1-\theta)} = b^{ka}.
\end{aligned}$$

The second integral from Holder's inequality can be computated directly as a geometric series. With with C a constant depending only on $A, q, s,$ and d, and

$d > s\zeta_+ 1 - \frac{1}{q}$, we have

$$\begin{aligned}
\int_{B_j^c} \rho(u)^{q'(s\zeta_+ - d)} du &= \sum_{m=j}^{\infty} \int_{B_{m+1}\setminus B_m} \rho(u)^{q'(s\zeta_+ - d)} du \\
&= (b-1) \sum_{m=j}^{\infty} b^{m(1+q'(s\zeta_+ - d))} \\
&= (b-1) b^{j(1+q'(s\zeta_+ - d))} \sum_{m=0}^{\infty} b^{m(1+q'(s\zeta_+ - d))} \\
&= C b^{j(1+q'(s\zeta_+ - d))}.
\end{aligned}$$

Therefore,

$$\left(\int_{B_j^c} \rho(u)^{q'(s\zeta_+ - d)} du \right)^{1/q'} = C b^{j(1-\frac{1}{q}+s\zeta_+ - d)}.$$

This gives

$$\left| \int_{B_j} M(u) \overline{Q_\alpha(A^{-j}u)} du \right| \leq C b^{-js\zeta_+} b^{ka} b^{j(1-\frac{1}{q}+s\zeta_+ - d)} = C b^{ka} b^{j(1-\frac{1}{q}-d)}.$$

Going back to $\pi_{B_j} M$, we have

$$\begin{aligned}
\|\pi_{B_j} M\|_{L^q} &\leq C b^{-j} b^{ka} b^{j(1-\frac{1}{q}-d)} b^{\frac{j}{q}} = C b^{-jd} b^{-k(-a)} \\
&= C b^{j(\frac{1}{q}-\frac{1}{p})} b^{j(\frac{1}{p}-\frac{1}{q})} b^{-jd} b^{-k(-a)} = C b^{j(\frac{1}{q}-\frac{1}{p})} b^{(j-k)(-a)}.
\end{aligned}$$

Finally, going back to $g_{j+1} - g_j$, we have

$$\begin{aligned}
\|g_{j+1} - g_j\|_q &\leq \|M \mathbf{1}_{B_{j+1}\setminus B_j}\|_q + \|\pi_{B_{j+1}} M\|_q + \|\pi_{B_j} M\|_q \\
&\leq C |B_{j+1}|^{\frac{1}{q}-\frac{1}{p}} b^{(j-k)(-a)}.
\end{aligned}$$

Therefore if $j > k$, $g_{j+1} - g_j = \mu_j a_j$, with $\mu_j = Cb^{(j-k)(-a)}$ and where a_j is a (p, q, s) atom supported on B_{j+1}. Summing the coefficients, we have

$$\sum_{j=k}^{\infty} |\mu_j|^p = \mu_k + \sum_{j=1}^{\infty} C^p b^{-jap} = (1 + C_0) + \frac{C}{1 - b^{-ap}}.$$

All together, we have shown that a (p, q, d) molecule is also in $H^p(\mathbb{R}^n)$, with

$$\|M\|_{H^p} \leq CN(M),$$

with C depending only on A, p, q, s, d and the cube Q, and is independent of M. This completes our proof of Theorem (2.3). $\square$

CHAPTER III

ANISOTROPIC MULTIPLIER THEOREMS

We now extend the theorems of Taibleson and Weiss [TW80] and Baernstein and Sawyer [BS85] to the anisotropic setting. Aside from these two results, there are other multiplier conditions in both the isotropic (classical) and anisotropic setting. In the isotropic setting, we can define the Mihlin condition of order s if $s = \left\lfloor n(\frac{1}{p} + \frac{1}{2}) \right\rfloor + 1$, and $m \in C^s(\mathbb{R}^n \backslash \{0\})$ has the property that for all γ such that $|\gamma| \leq s$,

$$|\partial^\gamma m(\xi)| \leq C_\gamma |\xi|^{-|\gamma|}.$$

This pointwise Mihlin condition implies Hörmander's condition (1.6) and is strong enough for the associated multiplier operator to be bounded on Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}$ (see Peetre [Pee75]). Table 3.1 summarizes the three multiplier conditions, in the order that increases the class of multipliers.

TABLE 3.1. Classical Multiplier Theorems

Condition	Mihlin of order s	$\Rightarrow$	Hörmander of order s	$\Leftrightarrow$	$K_2^{s,2}$	$\Rightarrow$	$K_1^{n(\frac{1}{p}-1),p}$
T_m bounded on	$\dot{B}_p^{\alpha,q}$ and $\dot{F}_p^{\alpha,q}$		H^p		H^p		H^p
Reference	[Pee75]		[TW80]		[CT75]		[BS85]

The arrows in the first row mean the Mihlin condition of order s implies the Hörmander condition of order s, which is equivalent to the Herz condition $K_2^{s,2}$, which implies the Herz condition $K_1^{n(\frac{1}{p}-1),p}$, making the last condition the most general.

In the anisotropic setting, Ding and Lan [DL06] were the first to study the multiplier problem, obtaining multiplier theorems associated with expansive symmetric matrices, extending the argument of [BS85]. Our Theorem 3.2 removes the requirement that the matrices be symmetric. Benyi and Bownik [BB10] used the anisotropic Mihlin condition on bounded symbols associated with pseudodifferential operators. Our Theorem 3.1 is closely related to their result, though we require minimal regularity requirement on m.

In an attempt to adapt the classical proof to the anisotropic setting, we run into significant problems. Theorem 1.7 of Taibleson and Weiss uses molecules, which can be defined in both time and frequency domains by the Fourier transform. This is possible due to Parseval's identity and the relationship between smoothness and decay under the Fourier transform: For α, there exists a constant C depending only on n and α such that

$$\int |f(x) \cdot x^{\alpha}|^2 dx \simeq \int |\partial^{\alpha} \hat{f}(\xi)|^2 d\xi.$$

In the anisotropic setting, the generalization of the Euclidean norm $|x|$ to the quasinorm $\rho(x)$ destroys much of this symmetry, and more care is needed to relate decay (now measured by ρ) to smoothness (still measured by ∂^{α}).

Another hurdle is the reliance on the *chain rule*: For $j \in \mathbb{Z}$,

$$\partial^{\alpha}(f(2^j x)) = 2^{j|\alpha|}(\partial^{\alpha} f)(2^j(x)).$$

But if we replace $2^j x$ by Ax where A is a matrix, then we do not have a usable *algebraic* formula to carry out our analysis. We can get around this if we know, roughly speaking, that A has small operator norm.

The main hurdle regarding the theorems of Baernstein and Sawyer is their use of the characterization of classical H^p in terms of Riesz transforms (due to Stein and Weiss [SW68]) in extending the boundedness of $T : H^p \to L^p$ to $T : H^p \to H^p$. But this singular integral characterization of H^p is not available in the anisotropic setting, as there are no analogues of the Riesz transforms.

Because of these considerations, we will only be able to adapt some of the results in the isotropic setting. Specifically, we are unable to reinterpret molecules in the frequency domain. Instead, we start with a multiplier m and study $K = \check{m}$. This analysis will provide the first multiplier theorem. The second multiplier theorem is restricted to the setting of $T : H^p_A \to L^p$ due to the the aforementioned difficulty of obtaining a singular integral characterization of H^p.

3.1. Two Anisotropic Multiplier Theorems

We now present the two multiplier theorems. We first observe that if A^* is the adjoint of A, then A^* is also a dilation matrix with its own anisotropic ellipsoids $\{B^*_j\}$. The anisotropic Mihlin condition first appeared in [BB10]. Since we will be working in both the time and frequency domain, we will generally reserve ξ for the independent variable in the frequency domain, and ∂_ξ denotes differentiation with respect to ξ.

Definition 3.1. Let $N \in \mathbb{N} \cup \{0\}$ and let $m \in C^N(\mathbb{R}^n \backslash \{0\})$. We say m has the anisotropic Mihlin condition of order N if there exists a constant $C = C_N$ such that for all multi-indices β such that $|\beta| \leq N$, all $j \in \mathbb{Z}$, and all $\xi \in B^*_{j+1} \backslash B^*_j$,

$$|D^{-j}_{A^*} \partial^\beta_\xi D^j_{A^*} m(\xi)| \leq C. \tag{3.1}$$

Theorem 3.1. *Let $N \in \mathbb{N}$ and suppose m satisfies the Mihlin condition of order N and $T_m : H^p_A(\mathbb{R}^n) \to H^p_A(\mathbb{R}^n)$ is the corresponding multiplier operator, and denote $L = \left(N\frac{\log\lambda_-}{\log b} - 1\right)\frac{\log b}{\log\lambda_+}$. Then T_m is bounded on H^p_A provided*

$$0 \leq \frac{1}{p} - 1 < \lceil L-1 \rceil \frac{(\log\lambda_-)^2}{\log b \log\lambda_+}.$$

For our second multiplier theorem, we introduce the anisotropic Herz spaces. Recall for a fixed annulus $\{B_j\}_{j\in\mathbb{Z}}$, the associated annuli are given by $\mathcal{A}_j = B_{j+1}\backslash B_j$ for all $j \in \mathbb{Z}$.

Definition 3.2. Let $a \in [1,\infty]$, let $s \in [0,\infty)$, and let $\beta \in (0,\infty]$. If $f \in L^a_{loc}(\mathbb{R}^n\backslash\{0\})$, we define the anisotropic Herz norm by

$$\|f\|^{\beta}_{\dot{K}^{s,\beta}_a} = \sum_{k=-\infty}^{\infty}\left(\int_{\mathcal{A}_k}|f(x)|^a dx\right)^{\frac{\beta}{a}} b^{ks\beta}, \tag{3.2}$$

and define

$$\|f\|_{K_{a^{s,\beta}}} = \|f\|_{L^a} + \|f\|_{\dot{K}^{s,\beta}_a}. \tag{3.3}$$

Let $w : \mathbb{N}\cup\{0\} \to [1,\infty)$ be an increasing sequence, that is, $w(k) \leq w(k+1)$ for all k. For $f \in L^1(\mathbb{R}^n)$, we define

$$\|f\|_{K(w)} = \int_{B_1}|f(x)|dx + \sum_{k=1}^{\infty}\left(\int_{\mathcal{A}_k}|f(x)|dx\right)w(k). \tag{3.4}$$

Then $K(w) = \{f \in L^1(\mathbb{R}^n) : \|f\|_{K(w)} < \infty\}$ is the weighted Herz space.

We define $\mathcal{A}_k = B_{k+1}\backslash B_k$ the k^{th} annulus. We fix $\psi \in S$ such that $\hat{\psi} \in C^\infty_c(\mathbb{R}^n)$, is supported on $B^*_1\backslash B^*_{-1}$, with values in $[0,1]$, and such that for all $\xi \neq 0$, we have

$\sum_{j\in\mathbb{Z}} \hat{\psi}(A^{*-j}\xi) = 1$. We also fix $\eta \in S$ such that $\hat{\eta} \in C_c^\infty(\mathbb{R}^n)$, supported on $B_2^* \backslash B_{-2}^*$, with values in $[0,1]$, such that on $B_1^* \backslash B_{-1}^*$, we have $\hat{\eta}(\xi) = 1$. Trivially, $\hat{\eta}\hat{\psi} = \hat{\psi}$. Let a be a $(p,2,s)$ atom supported on the ellipsoid B_1, and define $f = (m\hat{a})^\vee$, and

$$m_j(\xi) = m(A^{*j}\xi)\hat{\eta}(\xi), \qquad f_j = (m_j)^\vee, \tag{3.5}$$

$$\hat{a}_j(\xi) = \hat{a}(A^{*j}\xi)\hat{\psi}(\xi), \qquad g_j = (\hat{a}_j)^\vee. \tag{3.6}$$

Recall the identity $b^j \mathcal{F} D_{A^j}(f_j * g_j)(\xi) = m(\xi)\eta(A^{*-j}) \cdot \hat{a}(\xi)\psi(A^{*-j})$. Then we have

$$(m\hat{a})^\vee(x) = \sum_{j\in\mathbb{Z}} b^j (f_j * g_j)(A^j x). \tag{3.7}$$

Theorem 3.2. *Let m be a measurable function and let $T_m f = (\hat{f}m)^\vee$ be its multiplier operator.*

1. *Let $p \in (0,1)$. Then $T_m : H_A^p \to L^p$ is bounded provided m satisfies*

$$\sup_{j\in\mathbb{Z}} \|(m_j)^\wedge\|_{K_1^{\frac{1}{p}-1,p}} < \infty. \tag{3.8}$$

2. *Let $w : \mathbb{N}\cup\{0\} \to [1,\infty)$ be an increasing sequence satisfying $\displaystyle\sum_{k=0}^\infty \frac{1}{w(k)^2} < \infty$. Then $T_m : H_A^1 \to L^1$ is bounded provided m satisfies*

$$\sup_{j\in\mathbb{Z}} \|(m_j)^\wedge\|_{K(w)} < \infty. \tag{3.9}$$

3.1.1. Proof of Theorem (3.1)

We now prove Theorem (3.1). Unlike the classical multiplier theorem of Taibleson and Weiss, most of our analysis will take place in the time domain. Starting with m, we will show that $K = \check{m}$ has enough regularity so that $Tf = K * f$ is bounded on H^p_A.

Definition 3.3. Let (p, q, s) be admissible and let $R \in \mathbb{N}$ satisfy

$$R > \max\left\{ \left(\frac{1}{p} - 1\right) \frac{\log b}{\log \lambda_-}, s\frac{\log \lambda_+}{\log \lambda_-} \right\}, \tag{3.10}$$

and let $K \in \mathbb{C}^R(\mathbb{R}^n \backslash \{0\})$. We say K is a Calderón-Zygmund convolution kernel of order R if there exists a constant C such that for all multi-indices α with $|\alpha| \leq R$, and all $k \in Z$, $x \in B_{k+1} \backslash B_k$,

$$|D_A^{-k} \partial_x^\alpha D_A^k K(x)| \leq \frac{C}{\rho(x)}. \tag{3.11}$$

If K is such a kernel, we say K satisfies (CZC-R) and its associated singular integral operator T is defined by $Tf = K * f$.

The following lemma first appears as Theorem 9.8 of [Bow03] for the more general Calderón-Zygmund operators. We give an alternate proof using the molecular decomposition of H^p_A.

Lemma 3.1. *Let $R \in \mathbb{N}$. Suppose T is a singular integral operator whose kernel K is a Calderón-Zygmund convolution kernel of order R. Then $T : H^p_A \to H^p_A$ is bounded provided p satisfies*

$$0 < \frac{1}{p} - 1 < R\left(\frac{(\log \lambda_-)^2}{\log b \log \lambda_+}\right). \tag{3.12}$$

Proof. Let p satisfy the above inequality and (p, q, s) is an admissible triple. We will show there exists a constant C such that for all (p, q, s) atoms a, Ta is a molecule with $N(Ta) \leq C$. Once this uniform bound is established, Lemma 2.3

implies $\|Ta\|_{H^p_A} \le C$, which establishes the boundedness of T on H^p_A. To bound the molecular norm $N(Ta) = \|Ta\|_q^{1-\theta}\|\rho(x-x_0)^d Ta(x)\|_q^{\theta}$, we first note that as a singular integral operator, T is bounded from L^q to L^q for $q > 1$. There is a C, depending only on T, q, and θ, such that

$$\|Ta\|_q^{1-\theta} \le C\|a\|_q^{1-\theta} \le Cb^{r(\frac{1}{q}-\frac{1}{p})(1-\theta)},$$

where $C = C(T, q, \theta)$. By Minkowski's inequality:

$$\begin{aligned}\|\rho(x-x_0)^d Ta(x)\|_q \le & \left(\int_{x_0+B_{r+2\omega}} |\rho(x-x_0)^{dq} Ta(x)|^q dx\right)^{1/q} \\ & + \left(\int_{(x_0+B_{r+2\omega})^c} |\rho(x-x_0)^{dq} Ta(x)|^q dx\right)^{1/q} = I_1 + I_2\end{aligned}$$

where we denote the two integrals by I_1 and I_2, respectively. The estimate for I_1 is immediate:

$$I_1 \le b^{d(r+2\omega)} \left(\int_{x_0+B_{r+2\omega}} |Ta(x)|^q dx\right)^{1/q} \le b^{d(r+2\omega)}\|Ta\|_q \le C^{\frac{1}{1-\theta}} b^{dr} b^{r(\frac{1}{q}-\frac{1}{p})} = C^{\frac{1}{1-\theta}} b^{r(d+\frac{1}{q}-\frac{1}{p})}.$$

To estimate I_2, we require the following pointwise estimate from [Bow03, Lemma 9.5]: Suppose T is a singular integral operator whose kernel k is (CZC-R), with R satisfying (3.10). Then there exists a constant C such that for every (p, q, s) atom a with support $x_0 + B_r$, all $l \ge 0$ and $x \in x_0 + (B_{r+l+2\omega+1} \backslash B_{r+l+2\omega})$,

$$|Ta(x)| \le Cb^{-lR\zeta_- -l}|B_r|^{-1/p}. \tag{3.13}$$

With this estimate, we have

$$
\begin{aligned}
I_2 &= \sum_{j=0}^{\infty} \int_{x_o+(B_{r+2\omega+j+1}\setminus B_{r+2\omega+j})} \rho(x-x_o)^{dq}|Ta(x)|^q dx \\
&\le Cb^{-\frac{rq}{p}} \sum_{j=0}^{\infty} b^{-jq(1+R\zeta_-)} \left(\int_{x_o+B_{r+2\omega+j+1}\setminus B_{r+2\omega+j}} \rho(x-x_o)^{dq} dx \right) \\
&= Cb^{-\frac{rq}{p}} \sum_{j=0}^{\infty} b^{-jq(1+R\zeta_-)} b^{(dq+1)(r+2\omega+j)} = Cb^{r(dq+1-\frac{q}{p})} \sum_{j=0}^{\infty} b^{j(dq+1-q(1+R\zeta_-))}.
\end{aligned}
$$

The geometric series converges exactly when R satisfies (3.12). Taking the power θ/q on both sides, we have

$$
\|\rho(x-x_o)^d Ta(x)\|_q^\theta \le Cb^{r\theta(d+\frac{1}{q}-\frac{1}{p})}.
$$

Going back to the molecular norm,

$$
N(Ta) \le Cb^{k(\frac{1}{q}-\frac{1}{p})(1-\theta)} b^{k\theta(d+\frac{1}{q}-\frac{1}{p})} = C,
$$

as the exponent is exactly 0. $\square$

Theorem 3.1 follows from the following lemma that relates the relationship between m and $K = \check{m}$.

Lemma 3.2. *Suppose m satisfies the Mihlin condition of order N and let $K = \check{m}$. Then K is a Calderón-Zygmund convolution kernel of order R provided $R \in \mathbb{N}$ satisfies*

$$
0 \le R < \left(N \frac{\log \lambda_-}{\log b} - 1 \right) \frac{\log b}{\log \lambda_+}. \tag{3.14}
$$

Proof. Let m satisfy the Mihlin condition of order N and let R satisfy (3.14). We will show that $K = \check{m}$ is the kernel of a Calderón-Zygmund operator of order R. Fix $\Psi \in S(\mathbb{R}^n)$ such that $\hat{\Psi}$ is supported on $B_1^* \backslash B_{-1}^*$, and for all $\xi \neq 0$,

$$\sum_{j \in \mathbb{Z}} \hat{\Psi}(A^{-j}\xi) = 1.$$

By setting $\Psi_j(x) = b^j \Psi(A^j x)$, we have the identity $\widehat{\Psi_j}(\xi) = D_{A^*}^{-j} \hat{\Psi}(\xi) = \hat{\Psi}((A^*)^{-j})\xi)$, and $\widehat{\Psi_j}$ is supported on $B_{j+1}^* \backslash B_{j-1}^*$. We define $m_j = m\widehat{\Psi}_j$, which is supported on $B_{j+1}^* \backslash B_{j-1}^*$, and define $K_j = (m_j)^\vee$. Then we have immediate convergences:

$$\begin{aligned} m &= \sum_{j \in \mathbb{Z}} m_j \text{ pointwise and in } S', \\ K &= \sum_{j \in \mathbb{Z}} K_j \text{ in } S'. \end{aligned}$$

We'll see that the equality for K also holds pointwise.

We make the following reductions to prove the (CZC-R) condition (3.11). First, it suffices to show that for all β such that $|\beta| \leq R, k \in \mathbb{Z}, x \in B_1 \backslash B_0$,

$$|\partial_x^\alpha D_A^k K(x)| \leq \frac{C}{b^k}.$$

Second, this will follow from the absolute convergence

$$\sum_{j \in \mathbb{Z}} |\partial_x^\beta D_A^k K_j(x)| \leq \frac{C}{b^k}.$$

Lastly, it suffices to prove the above convergence for $k = 0$:

$$\sum_{j \in \mathbb{Z}} |\partial_x^\beta K_j(x)| \leq C. \tag{3.15}$$

Indeed, if $k \in \mathbb{Z}$, and m has the Mihlin property, then so does $D_{A^*}^k m$, with the same constant C. Therefore if $\xi \in B_{j+1}^* \backslash B_j^*$, then $A^{*k}\xi \in B_{j+k+1} \backslash B_{j+k}$.

$$|(D_{A^*}^{-j}\,\partial_\xi^\beta\, D_{A^*}^j)(D_{A^*}^k m)(\xi)| = |(D_{A^*}^{-j-k}\,\partial_\xi^\beta\, D_{A^*}^{j+k} m(A^{*k}\xi)| \leq C_\beta.$$

To prove (3.15), we decompose the sum using a well-chosen integer M, which will depend on how close R is to L. Denote $\lambda_{\max}^*$ as the eigenvalue of A^* with the largest norm and $\|\cdot\|_{\text{op}}$ is the operator norm on $\mathbb{R}^n \to \mathbb{R}^n$. By the spectral theorem,

$$\lambda_{\max}^* = \limsup_{j \to \infty} \|A^{*j}\|_{\text{op}}^{1/j}.$$

Let $\epsilon > 0$. Then there exists an integer $M > 0$ such that for all $j > M$,

$$\|A^{*j}\|_{\text{op}}^{1/j} \leq (1+\epsilon)\lambda_{\max}^* \leq (1+\epsilon)\lambda_+.$$

With this M, we write

$$\sum_{j \in \mathbb{Z}} |\partial_x^\beta K_j(x)| = \sum_{j \leq M} |\partial_x^\beta K_j(x)| + \sum_{j > M} |\partial_x^\beta K_j(x)| = S_L + S_H.$$

We call S_L and S_H the low and high spatial terms, respectively. Starting with the high spatial terms, we fix $j > M$ and $x \in B_{-1} \backslash B_0$. Then we can fix another multi-index α satisfying $|\alpha| = N$ such that there exists a constant c depending only on n such that $|(A^j x)^\alpha| \geq c|A^j x|^N$. This can be done by picking $\alpha = Ne_i$ where e_i is the

i^{th} unit vector in the canonical basis of $\mathbb{R}^n$ and the direction i is where $A^j x$ has the largest value in norm. Define $w(u) = (A^{*j}u)^\beta m(A^{*j}u)\hat{\Psi}(u)$. Using Parseval's identity, integration by parts, and a change of variables, we have

$$\partial_x^\beta K_j(x) = c_\beta b^j \int_{B_1^* \backslash B_{-1}^*} (\partial_u^\alpha w)(u) \frac{e^{2\pi i \langle A^j x, u\rangle}}{(2\pi i A^j x)^\alpha} du,$$

which we estimate using the bound from the spectral theorem.

If $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\gamma = (\gamma_1, \ldots, \gamma_n)$ are two multi-indices, we say $\gamma \leq \alpha$ if for each i, $1 \leq i \leq n$, $\gamma_i \leq \alpha_i$. We also define the binomial coefficients by

$$\binom{\alpha}{\gamma} = \binom{\alpha_1}{\gamma_1} \cdots \binom{\alpha_n}{\gamma_n}.$$

Then the product rule gives:

$$(\partial^\alpha w)(u) = \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} \underbrace{\partial^{\alpha-\gamma}(D_{A^*}^j m \cdot \hat{\Psi})(u)}_{I_1} \cdot \underbrace{\partial^\gamma((A^{*j}u)^\gamma)}_{I_2}. \tag{3.16}$$

By another application of the product rule, we have a uniform constant c', independent of m, j, u such that

$$I_1 \leq c \sup_{\delta \leq \gamma} |\partial^\delta D_{A^*}^j m)(u)| = c \sup_{\delta \leq \gamma} |(D_{A^*}^{-j} \partial^\delta D_{A^*}^j m)(A^{*j}u)| \leq c'.$$

We now bound I_2. With $u \in B_1^* \backslash B_{-1}^*$, elementary considerations from expressing $(A^{*j}u)^\beta$ as a sum of monomials show that there exists c depending only on N, such that by our choice of M and $j > M$,

$$I_2 = |\partial^\gamma (A^{*j}u)^\beta| \leq c \|A^{*j}\|_{\text{op}}^{|\beta|} \leq c(\lambda_+^*(1+\epsilon))^{j|\beta|},$$

Combining our estimates of I_1 and I_2 in (3.16), we have a constant C, depending on the past constants, such that

$$|(\partial^\alpha w)(u)| \leq C(\lambda_+^*(1+\epsilon))^{j|\beta|}.$$

Then we have

$$\begin{aligned} |\partial^\beta K_j(x)| &\leq b^j \int_{B_1^* \setminus B_{-1}^*} \left| \frac{(\partial^\alpha w)(u)}{(2\pi i A^j x)^\alpha} \right| du \\ &\leq C \left(\frac{b^j (\lambda_+^*(1+\epsilon))^{j|\beta|}}{|A^j x|^{|\alpha|}} \right) \leq C \left(\frac{b^j (\lambda_+^*(1+\epsilon))^{j|\beta|}}{b^{j|\alpha|\zeta_-}} \right). \end{aligned}$$

Note that with our choice of α and (1.11), we can sum $|\partial^\beta K_j(x)|$ for $j > M$ if

$$\frac{b(\lambda_+^*(1+\epsilon))^{|\beta|}}{b^{|\alpha|\zeta_-}} < 1 \qquad \text{that is,} \qquad |\beta| < \left(\frac{N \log \lambda_-}{\log b} - 1 \right) \frac{\log b}{\log(\lambda_+^*(1+\epsilon))}.$$

Indeed, for $|\beta| \leq R$, there exists $\epsilon > 0$ such that the series below converges: For C_1 depending only on A, n, Ψ, β, M, we have

$$\sum_{j=M+1}^{\infty} |\partial^\beta K_j(x)| \leq C \sum_{j=M+1}^{\infty} \left(\frac{b(\lambda_+^*(1+\epsilon))^{|\beta|}}{b^{|\alpha|\zeta_-}} \right)^j \leq C_1.$$

Turning our attention to S_L, we start with Parseval's identity and a change of variables. With C a dimensional constant, we have

$$\begin{aligned} |\partial_x^\beta K_j(x)| &= \left| \int_{B_{j+1}^* \setminus B_{j-1}^*} (2\pi i \xi)^\beta m_j(\xi) e^{2\pi i \langle x, \xi \rangle} d\xi \right| \\ &\leq cb^j \int_{B_1^* \setminus B_{-1}^*} \underbrace{|(A^{*j} u)^\beta|}_{J_1} \cdot \underbrace{|m_j(A^{*j} u)|}_{J_2} du \leq \begin{cases} Cb^{j(1+|\beta|\zeta_+)} & \text{if } j \geq 0 \\ Cb^{j(1+|\beta|\zeta_-)} & \text{if } j < 0. \end{cases} \end{aligned}$$

Indeed, for u in the unit annulus and $A^{*j}u \in B^*_{j+1} \backslash B^*_{j-1}$, $J_1 \leq c|A^{*j}u|^{|\beta|} \leq Cb^{j\zeta_\pm}\rho_*(u)^{\zeta_\pm}$, with the eccentricity $\zeta_\pm$ depending on the sign of j. Since $m \in L^\infty$ and $J_2 \leq C(m, \Psi)$, we obtain the above estimate. Returning to S_L, we have a constant C, depending only on n, A, N, Ψ, M such that

$$S_L \leq \sum_{j=-\infty}^{-1} |\partial_x^\beta K_j(x)| + \sum_{j=0}^{M} |\partial_x^\beta K_j(x)| \leq C \sum_{j=-\infty}^{-1} b^{j(1+|\beta|\zeta_-)} + C \sum_{j=0}^{M} b^{j(1+|\beta|\zeta_+)} \leq C_2,$$

with $C_2 = C_2(n, A, N, \Psi, M)$. This completes the estimate (3.15), and the proof of this theorem. $\square$

Proof of Theorem 3.1. Let m be satisfy the Mihlin condition of order N. Recall that we defined

$$L = \left(N \frac{\log \lambda_-}{\log b} - 1 \right) \frac{\log b}{\log \lambda_+}.$$

If L is not an integer, $\lceil L \rceil - 1 < L$. By Lemma (3.2), $K = \check{m}$ is a (CZC-R) kernel with $R = \lceil L \rceil - 1$. Using this value of R in Lemma 3.1, we obtain the first range of p on which $T : H^p_A \to H^p_A$ is bounded. Similarly, if L is an integer, then we take $R = L - 1$. This gives the second range of p. $\square$

3.2. Proof of Theorem 3.2

We need two lemmas that provide estimates for g_j. We will defer the proofs to the last section.

Lemma 3.3. *Let $r > 0$.*

1. *There exists a constant C_1, depending only on A, s, ψ, r, such that for all $j \leq 0$ and $x \in \mathbb{R}^n$,*

$$|g_j(x)| \leq C_1 \frac{b^{j(s+1)\zeta_-}}{(1 + \rho(x))^r}.$$

2. There exists a constant C_2, depending only on A, p, r, such that for all $j \geq 0$,

$$|g_j(x)| \leq \begin{cases} C_2 b^{-j/2} & \text{for all } x, \\ C_2 \dfrac{b^{-j/2}}{\rho(x)^r} & \text{for } x \in B_{j+1}^c. \end{cases}$$

Lemma 3.4. *Let $p \in (0,1)$, let $j \geq 0$, let $r > \frac{1}{p}$, and $g \in K_1^{\frac{1}{p}-1,p}$. Then there exists a constant C_3, depending only on A, r, p, such that if $Q \in L^1(\mathbb{R}^n)$ satisfies $\|Q\|_{L^1} \leq 1$ and $|Q(x)| \leq \dfrac{1}{\rho(x)^r}$ if $x \in B_{j+1}^c$, then*

$$\sum_{k=j+2\omega}^{\infty} \left(\int_{\mathcal{A}_k} |(g * Q)(x)| dx \right)^p b^{k(1-p)} \leq C_3 \|g\|^p_{K_1^{(\frac{1}{p}-1),p}}.$$

We make one more important observation. By a simple application of Hölder's inequality, there is a constant C, depending only on b and p, such that

$$\|f\|_{L^p}^p \leq \sum_{k\in\mathbb{Z}} \left(\int_{\mathcal{A}_k} |f(x)| dx \right)^p \left(\int_{\mathcal{A}_k} dx \right)^{1-p} = C \|f\|^p_{\dot{K}_1^{\frac{1}{p}-1,p}}.$$

Then we have

$$\|f\|_{L^p} \leq C(p,b) \|f\|_{\dot{K}_1^{\frac{1}{p}-1,p}}. \tag{3.17}$$

Proof of Theorem 3.2 for $p < 1$. In light of (3.17), it suffices to prove the following estimate for all unit $(p,2,s)$ atoms a, that is, atoms supported on B_1:

$$\|T_m a\|^p_{\dot{K}_1^{\frac{1}{p}-1,p}} = \sum_{k\in\mathbb{Z}} \left(\int_{\mathcal{A}_k} |(m\hat{a})^\vee(x)| dx \right)^p b^{k(1-p)} \leq C. \tag{3.18}$$

To simplify our proof, we may assume $\sup_{j\in\mathbb{Z}} \|(m_j)^\wedge\|_{K_1^{\frac{1}{p}-1,p}} = 1$. By the definition, for all $j \in \mathbb{Z}$, we have

$$\|(m_j)^\wedge\|_{K_1^{\frac{1}{p}-1,p}} = \|(m_j)^\wedge\|_{L^1} + \|(m_j)^\wedge\|_{\dot{K}_1^{\frac{1}{p}-1,p}} \leq 1.$$

We split the sum in (3.18) into the cases $k \leq 2\omega - 1$ and $k \geq 2\omega$. If $k \leq 2\omega - 1$, there is a constant $C_{b,m}$, depending only on b and m, such that by Hölder's inequality,

$$\begin{aligned}\int_{B_{2\omega}} |(m\hat{a})^\vee(x)|dx &\leq \left(\int_{B_{2\omega}} |(m\hat{a})^\vee(x)|^2 dx\right)^{1/2} |B_{2\omega}|^{1/2} \\ &\leq b^\omega \left(\int_{\mathbb{R}^n} |m(\xi)\hat{a}(\xi)|^2 d\xi\right)^{1/2} \leq C_{b,m}.\end{aligned}$$

Then we have $C = C(A, p, m)$ so that

$$\sum_{k=-\infty}^{2\omega} \left(\int_{\mathcal{A}_k} |(m\hat{a})^\vee(x)|dx\right)^p b^{k(1-p)} \leq C \sum_{k=-\infty}^{2\omega} b^{k(1-p)} \leq \frac{C}{1 - b^{1-p}}.$$

Now fix $k \geq 2\omega$. We use (3.7) to decompose the integral in the k^{th} term of the sum.

$$\begin{aligned}\int_{\mathcal{A}_k} |(m\hat{a})^\vee(x)|dx &\leq \sum_{j=-\infty}^{\infty} \int_{\mathcal{A}_{k+j}} |(f_j * g_j)(y)|dy \\ &= \sum_{j=-\infty}^{-k+2\omega-1} + \sum_{j=-k+2\omega}^{0} + \sum_{j=1}^{\infty} \int_{\mathcal{A}_{k+j}} |(f_j * g_j)(y)|dy\end{aligned}$$

We denote the three sums by α_k, β_k, and γ_k respectively. Summing over k,

$$\begin{aligned}&\sum_{k=2\omega}^{\infty} \left(\int_{\mathcal{A}_k} |(m\hat{a})^\vee(x)|dx\right)^p b^{k(1-p)} \leq \sum_{k=2\omega}^{\infty} (\alpha_k^p + \beta_k^p + \gamma_k^p)\, b^{k(1-p)} \\ &= \sum_{k=2\omega}^{\infty} \alpha_k^p b^{k(1-p)} + \sum_{k=2\omega}^{\infty} \beta_k^p b^{k(1-p)} + \sum_{k=2\omega}^{\infty} \gamma_k^p b^{k(1-p)} = S_\alpha + S_\beta + S_\gamma.\end{aligned}$$

To estimate S_α, for each k, we apply Part 1 of Lemma 3.3 to α_k, with $j \le 0$ and r large enough so that the integral $\displaystyle\int_{\mathbb{R}^n} \frac{1}{(1+\rho(x))^r} dx$ is finite. Then there exists a constant C_α, depending only on A and r so that

$$\begin{aligned}\alpha_k &= \sum_{j=-\infty}^{-k+2\omega} \int_{A_{k+j}} |(f_j * g_j)(y)| dy \le \sum_{j=-\infty}^{-k+2\omega} \|f_j\|_{L^1} \cdot \|g_j\|_{L^1} \le \sum_{j=-\infty}^{-k+2\omega} \|g_j\|_{L^1} \\ &\le C_\alpha \sum_{j=-\infty}^{-k+2\omega} \int_{\mathbb{R}^n} \frac{b^{j(s+1)\zeta_-}}{(1+\rho(x))^r} dx \le C_\alpha \sum_{j=-\infty}^{-k+2\omega} b^{j(s+1)\zeta_-} = C_\alpha b^{(-k+2\omega)(s+1)\zeta_-}.\end{aligned}$$

Summing over $k \ge 2\omega$, and by the fact that $(s+1)\zeta_- > \frac{1}{p} - 1$, we have a constant $C = C(A, p, s, r)$ so that

$$S_\alpha \le C_\alpha \sum_{k=2\omega}^{\infty} b^{p(-k+2\omega)(s+1)\zeta_-)+k(1-p)} \le C.$$

We estimate the sum S_γ:

$$\begin{aligned}S_\gamma &\le \sum_{k=2\omega}^{\infty} b^{k(1-p)} \sum_{j=1}^{\infty} \left(\int_{\mathcal{A}_{k+j}} |(f_j * g_j)(y)| dy \right)^p = \sum_{j=1}^{\infty} \sum_{l=j+2\omega}^{\infty} \left(\int_{\mathcal{A}_l} |f_j * g_j(y)| dy \right)^p b^{(l-j)(1-p)} \\ &= \sum_{j=1}^{\infty} \underbrace{\left[\sum_{l=j+2\omega}^{\infty} \left(b^{j/2} \int_{\mathcal{A}_l} |f_j * g_j(y)| dy \right)^p b^{l(1-p)} \right]}_{I_j} b^{-j(1-\frac{p}{2})}.\end{aligned}$$

To estimate I_j, observe that since $j \ge 1$ and $r > 0$, Part 2 of Lemma 3.3 gives $b^{j/2}|g_j(y)| \lesssim \frac{1}{\rho(y)^r}$ if $y \in B_{j+1}$. Up to a general constant c, we can apply Lemma 3.4 with $Q = cb^{j/2} g_j$ so $\|Q\|_{L^1} \le 1$, and using$f_j \in K_1^{\frac{1}{p}-1,p}$, we see that $I_j \le C \|f_j\|^p_{K_1^{\frac{1}{p}-1,p}}$. Then since $\|f_j\|_{K_1^{\frac{1}{p}-1,p}} \le 1$, and $1 - \frac{p}{2} > 0$, there exists a constant C_γ, depending only on A, r, p so that

$$S_\gamma \le C_\gamma \sum_{j=1}^{\infty} \|f_j\|^p_{K_1^{\frac{1}{p}-1,p}} \cdot b^{-j(1-\frac{p}{2})} \le C_\gamma.$$

Lastly, we estimate S_β.

$$S_\beta \le \sum_{k=2\omega}^{\infty} \sum_{j=-k+2\omega}^{0} \left(\int_{\mathcal{A}_{k+j}} |f_j * g_j(y)| dy \right)^p b^{k(1-p)} = \sum_{j=-\infty}^{0} \sum_{k=2\omega-j}^{\infty} \left(\int_{\mathcal{A}_{k+j}} |f_j * g_j| dy \right)^p b^{k(1-p)}$$
$$= \sum_{j=-\infty}^{0} b^{-j(1-p)} \underbrace{\sum_{\ell=2\omega}^{\infty} b^{\ell(1-p)} \left(\int_{\mathcal{A}_\ell} |f_j * g_j| dy b^{-j(s+1)\zeta_-} \right)^p}_{J_j} b^{jp(s+1)\zeta_-}.$$

We estimate J_j using Part 1 of Lemma 3.3 to obtain $b^{-j(s+1)\zeta_-}|g_j(y)| \lesssim \frac{1}{\rho(x)^r}$. Setting $Q = b^{j(s+1)\zeta_-} g_j$ in Lemma 3.4 gives a constant C_β, depending only on A, p, r such that $J_j \le C\|f_j\|^p_{K_1^{\frac{1}{p}-1,p}} \le C_\beta$. Inserting this into the above sum, we obtain another uniform bound C'_β, also dependent on A, p, r, so that $S_\beta \le C'_\beta$. This completes our estimate on S_β and our proof of (3.18). $\square$

Proof of Theorem 3.2 for $p = 1$. As in the case $p < 1$, it suffices to prove the following uniform bound for all $(1, 2, s)$ atoms a supported on B_1:

$$\|(m\hat{a})^\vee\|_{L^1} = \|f\|_{L^1} \le C. \tag{3.19}$$

We again make the following reductions. We suppose m satisfies

$$\sup_{j\in\mathbb{Z}} \|(m_j)^\wedge\|_{K(w)} = 1. \tag{3.20}$$

Then by our definition (3.4), our assumption that $w(k) \ge 1$so for all $j \in \mathbb{Z}$, and for all j, we have $\|m_j\|_{L^\infty} \le \|(m_j)^\vee\|_{L^1} \le 1$

$$\|f_j\|_{L^1} \le \|f_j\|_{K(w)} = \|(m_j)^\wedge\|_{K(w)} \le 1 \qquad \text{and} \qquad \|m_j\|_{L^\infty} \le 1. \tag{3.21}$$

In estimating $\|f\|_{L^1}$, we will repeatedly peel away 'low-spatial' terms, each of which is relatively simple to estimate, until we arrive at the term $\sum_{j=1}^{\infty}\|g_j\|^2_{L^1(B_j^c)}$. To start, $\|f\|_{L^1} = \|f\|_{L^1(B_{2\omega})} + \|f\|_{L^1(B_{2\omega}^c)}$, the first low-spatial term is the integral over $B_{2\omega}$. By Hölder's inequality, there is a constant C, depending on A, m, such that

$$\|f\|_{L^1(B_{2\omega})} \leq C\left(\int_{\mathbb{R}^n} |f(x)|^2 dx\right)^{1/2} = \left(\int_{\mathbb{R}^n} |m\hat{a}(\xi)|^2 d\xi\right)^{1/2} \leq C$$

To continue our estimate on the integral $\|f\|_{L^1(B_{2\omega}^c)}$, we use (3.7) to obtain the pointwise decomposition of f:

$$f(x) = \sum_{j=-\infty}^{0} b^j (f_j * g_j)(A^j x) + \sum_{j=1}^{\infty} b^j (f_j * g_j)(A^j x).$$

We denote the above two sums by f_L and f_H respectively. By Hölder's inequality,

$$\int_{B_{2\omega}^c} |f_L(x)| dx \leq \sum_{j=-\infty}^{0} \|f_j\|_{L^1}\|g_j\|_{L^1} \leq \sum_{j=-\infty}^{0} \|g_j\|_{L^1}.$$

Since $j \leq 0$, and taking $r > 1$, Part 1 of Lemma 3.3 gives a constant C, depending on C_1, r, A, s such that

$$\|g_j\|_{L^1} = \int_{\mathbb{R}^n} |g_j(x)| dx \leq C \int_{\mathbb{R}^n} \frac{b^{j(s+1)\zeta_-}}{(1+\rho(x))^r} dx = C b^{j(s+1)\zeta_-},$$

which implies

$$\int_{B_{2\omega}^c} |f_L(x)| dx \leq C \sum_{j=-\infty}^{0} \|g_j\|_{L^1} \leq C \sum_{j=-\infty}^{0} b^{j(s+1)\zeta_-} \leq C.$$

To continue our estimate for the integral over f_H, we give a pointwise estimate to the convolution $f_j * g_j$. For $k \in \mathbb{Z}$, we define two integers a_1, a_2, by

$$a_1(k) = \left\lfloor k - \omega - 1 + \frac{\log(b-1)}{\log b} \right\rfloor$$
$$a_2(k) = \max\{\lfloor k + \log(1+b^\omega)\rfloor, k + 2\omega\}.$$

Observe that $a_1 - a_k$ is a constant independent of k. Then

$$b^{-\omega} - b^{a_1 - k} \geq b^{-\omega-1} \tag{3.22}$$
$$b^{a_2 - k - \omega - 1} \geq b^{-\omega}. \tag{3.23}$$

By a change of variables, we have

$$\begin{aligned}\int_{B_{2\omega}^c} |f_H(x)| dx &\leq \sum_{j=1}^{\infty} \sum_{k=j+2\omega}^{\infty} \int_{\mathcal{A}_k} |(f_j * g_j)(y)| dy \\ &\leq \overbrace{\sum_{j=1}^{\infty} \sum_{k=j+2\omega}^{\infty} \left(\int_{B_{a_1}} |f_j(z) g_j(y-z)| dz + \int_{B_{a_2}^c} |f_j(z) g_j(y-z)| dz \right)}^{S_1} \\ &\quad + \underbrace{\sum_{j=1}^{\infty} \sum_{k=j+2\omega}^{\infty} \int_{B_{a_2} \setminus B_{a_1}} |f_j(z) g_j(y-z)| dz}_{S_2}.\end{aligned}$$

We start with S_1. Let $r > 0$, and fix $z \in B_{a_1}$ and $y \in B_k$. Then by (3.22), we can apply Lemma 3.3 on g_j to obtain

$$|g_j(y-z)| \lesssim \frac{1}{b^{(k-\omega-1)r}}.$$

For the integral over $B_{a_2^c}$, we fix and $z \in B_{a_2}^c, y \in B_k$. Then by Lemma 3.3 again, we have

$$|g_j(y-z)| \lesssim \frac{1}{b^{(k-\omega)r}}.$$

These estimates, combined with (3.21), give

$$|(f_j * g_j)(y)| \leq C(b^{-kr} + |(f_j \mathbf{1}_{B_{a_2} \setminus B_{a_1}} * |g_j|)(y)|).$$

Define $J(l,j) = \int_{\mathcal{A}_l} |f_j| dy$, with $a_2 - a_1$ depending only on ω, b and independent of k. Then integrating over $\mathcal{A}_k$, we have

$$\begin{aligned}\int_{\mathcal{A}_k} |(f_j * g_j)(y)| dy &\lesssim b^{-k(r-1)} + \int_{\mathcal{A}_k} \left(\int_{B_{a_2} \setminus B_{a_1}} |f_j(z) g_j(y-z)| dz \right) dy \\ &\leq b^{-k(r-1)} + \left(\int_{B_{a_2} \setminus B_{a_1}} |f_j(y)| dy \right) \|g_j\|_{L^1} = b^{-k(r-1)} + \sum_{l=a_1}^{a_2} J(l,j)\ \|g_j\|_{L^1}.\end{aligned}$$

Returning to S_1, we then have a constant C depending only on A, b, r, s and C_1 and C_2 from Lemma 3.3 and 3.4 such that

$$S_1 \leq \sum_{j=1}^{\infty} C b^{-j} + \sum_{k=j+2\omega}^{\infty} \sum_{l=a_1}^{a_2} J(l,j) \|g_j\|_{L^1} \leq C + C \sum_{j=1}^{\infty} \|g_j\|_{L^1} \sum_{k=j+2\omega}^{\infty} J(k,j).$$

Note that in the last inequality, we changed the index from $J(l,j)$ to $J(k,j)$, since l runs over a fixed range, regardless of what k is. We rewrite S_2 using the same reasoning:

$$S_2 = \sum_{k=j+2\omega}^{\infty} \sum_{j=1}^{\infty} \sum_{l=a_1}^{a_2} \int_{\mathcal{A}_l} |f_j(z) g_j(y-z)| dz \leq C \sum_{j=1}^{\infty} \|g_j\|_{L^1} \sum_{k=j+2\omega}^{\infty} J(k,j).$$

By the condition on our weight w and the definition of the Herz norm of f_j (3.4), we have

$$\begin{aligned}\sum_{k=j+2\omega}^{\infty} J(k,j) &= \sum_{k=j+2\omega}^{\infty} \frac{w(k)}{w(k)} J(k,j) \leq \frac{1}{w(j)} \sum_{k=j+2\omega}^{\infty} w(k) J(k,j) \\ &= \frac{1}{w(j)} \sum_{k=j+2\omega}^{\infty} \left(w(k) \int_{\mathcal{A}_k} |f_j| dy \right) \leq \frac{\|f_j\|_{K(w)}}{w(j)} \leq \frac{1}{w(j)}.\end{aligned}$$

In light of this, we can combine the estimates on S_1 and S_2 to obtain

$$\int_{B^c_{j+2\omega}} |f_H(x)| dx \leq S_1 + S_2 \leq C + C \left(\sum_{j=1}^{\infty} \|g_j\|^2_{L^1} \right)^{1/2} \|1/w\|_{\ell^2} \leq C + C \left(\sum_{j=1}^{\infty} \|g_j\|^2_{L^1} \right)^{1/2}.$$

We now prove a uniform estimate for the sum $\sum_{j=1}^{\infty} \|g_j\|^2_{L^1}$, starting with Hölder's inequality to obtain the following decomposition for each g_j:

$$\sum_{j=1}^{\infty} \|g_j\|^2_{L^1} = \underbrace{\sum_{j=1}^{\infty} \|g_j\|^2_{L^1(B_j)}}_{G_1} + \underbrace{\sum_{j=1}^{\infty} \|g_j\|^2_{L^1(B^c_j)}}_{G_2} + 2 \left(\sum_{j=1}^{\infty} \|g_j\|^2_{L^1(B_j)} \right)^{1/2} \left(\sum_{j=1}^{\infty} \|g_j\|^2_{L^1(B^c_j)} \right)^{1/2}.$$

We will prove that G_1 and G_2 are uniformly bounded. As has been the case in this proof, the estimate on the low-spatial sum G_1 is immediate. We recall that ψ is a smooth function defined on the annulus $B^*_1 \backslash B^*_{-1}$. By Hölder and Parseval's identity, we have a constant C depending only on ψ and A such that

$$\|g_j\|^2_{L^1(B_j)} \leq b^j \int_{B^*_1 \backslash B^*_{-1}} |\hat{a}((A^{*j})\xi)\psi(\xi)|^2 d\xi \leq C \int_{B^*_{j+1} \backslash B^*_{j-1}} |\hat{a}(u)|^2 du.$$

Summing over j, we have

$$G_1 = \sum_{j=1}^{\infty} \|g_j\|^2_{L^1(B_j)} \le C \sum_{j=1}^{\infty} \int_{B^*_{j+1}\setminus B^*_{j-1}} |\hat{a}(u)|^2 du \le 2C\|\hat{a}\|^2_{L^2},$$

which is uniformly bounded. It remains to estimate G_2. If we fix $j \ge 1$ and set $q = s\zeta_-$, we have a constant C depending only on A and q such that

$$\begin{aligned} \|g_j\|^2_{L^1(B_j^c)} &= \left(\int_{B_j^c} |g_j(x)| \, \frac{\rho(x)^q}{\rho(x)^q} dx \right)^2 \le \int_{B_j^c} |g_j(x)|^2 \rho(x)^{2q} dx \int_{B_j^c} \rho(x)^{-2q} dx \\ &\le C b^{j(1-2q)} \int_{B_j^c} |g_j(x)|^2 \rho(x)^{2q} dx. \end{aligned} \tag{3.24}$$

This last integral will be estimated by exploiting the fact that $\hat{g}_j$ has compact support in the frequency domain. By the same constant above, we use a change of variables and (2.9) and obtain

$$\begin{aligned} \int_{B_j^c} |g_j(x)|^2 \rho(x)^{2q} dx &= b^{j(2q-1)} \int_{B_0^c} |(b^j D_A^j g_j)(y)|^2 \rho(y)^{2q} dy \\ &\le C b^{j(2q-1)} \sum_{|\alpha|=s} \int_{\mathbb{R}^n} |\partial^\alpha (\mathcal{F} b^j D_A^j g_j)(\xi)|^2 d\xi \\ &= C b^{j(2q-1)} \sum_{|\alpha|=s} \int_{\mathbb{R}^n} |\partial^\alpha (D_{A^*}^{-j} \hat{g}_j)(\xi)|^2 d\xi, \end{aligned} \tag{3.25}$$

To motivate the expression in the present form, we first write:

$$\hat{g}_j(\xi) = \hat{a}((A^{*j})\xi) p\hat{s}i(\xi) \;\Rightarrow\; D_{A^*}^{-j} \hat{g}_j(\xi) = \hat{a}(\xi) \hat{\psi}(A^{*-j}\xi).$$

Applying the differential ∂^α, the product rule gives

$$\begin{aligned}|\partial^\alpha(D_{A^*}^{-j}\hat{g}_j)(\xi)|^2 &\leq \left(\sum_{\beta\leq\alpha}\left|\binom{\alpha}{\beta}\partial^\beta\hat{a}(\xi)\cdot\partial^{\alpha-\beta}[\hat{\psi}((A^{*-j})\xi)]\right|\right)^2\\ &\leq \left(\sum_{\beta\leq\alpha}\left|\binom{\alpha}{\beta}\partial^{\alpha-\beta}[\hat{\psi}(A^{*-j}\xi)]\right|^2\right)\left(\sum_{\beta\leq\alpha}|\partial^\beta\hat{a}(\xi)|^2\right)\\ &\leq C\left(\sup_{\gamma\leq\alpha}|\partial^\gamma[\hat{psi}((A^{*-j})\xi)]|^2\right)\sum_{\beta\leq\alpha}|\partial^\beta\hat{a}(\xi)|^2\\ &\leq C\mathbf{1}_{B_{j+1}^*\backslash B_{j-1}^*}(\xi)\sum_{\beta\leq\alpha}|\partial^\beta\hat{a}(\xi)|^2.\end{aligned}$$

Thus, the dilation operator is moved from $\hat{a}$ to ψ. Integrating over $\mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n}|\partial^\alpha(D_{A^*}^{-j}\hat{g}_j)(\xi)|^2d\xi \leq C\int_{B_{j+1}^*\backslash B_{j-1}^*}\sum_{\beta\leq\alpha}|\partial^\beta\hat{a}(\xi)|^2d\xi = C\sum_{\beta\leq\alpha}\int_{B_{j+1}^*\backslash B_{j-1}^*}|\partial^\beta\hat{a}(\xi)|^2d\xi. \tag{3.26}$$

Plugging (3.26) into (3.25), we have

$$\int_{B_j^c}|g_j(x)|^2\rho(x)^{2q}dx \leq C\sum_{|\alpha|=s}\sum_{\beta\leq\alpha}\int_{B_{j+1}\backslash B_{j-1}}|\partial^\beta\hat{a}(\xi)|^2d\xi. \tag{3.27}$$

Let C_n denote a dimensional constant. Since a is supported on B_1, the integral $\int_{B_1} |a(x)|^2 dx \le C_n$. We can now finish our estimate on G_2. By (3.24) and (3.27),

$$
\begin{aligned}
G_2 = \sum_{j=1}^{\infty} \|g_j\|^2_{L^1(B_j^c)} &\le C \ \sum_{j=1}^{\infty} \left(b^{j(1-2q)} \left[\int_{B_j^c} |g_j(x)|^2 \rho(x)^{2q} dx \right] \right) \\
&\le C \ \sum_{j=1}^{\infty} \left(\sum_{|\alpha|=s} \sum_{\beta \le \alpha} \int_{B_{j+1}^* \setminus B_{j-1}^*} |\partial^\beta \hat{a}(\xi)|^2 d\xi \right) \\
&\le C \sum_{j=1}^{\infty} \sum_{|\beta| \le s} \int_{B_{j+1}^* \setminus B_{j-1}^*} |\partial^\beta \hat{a}(\xi)|^2 d\xi \le C \sum_{|\beta| \le s} \int_{\mathbb{R}^n} |\partial^\beta \hat{a}(\xi)|^2 d\xi \\
&= C \sup_{|\beta| \le s} \int_{\mathbb{R}^n} |(a(x) \cdot x^\beta)|^2 dx \le C \sup_{m \le s} \int_{\mathbb{R}^n} |a(x)|^2 \ |x|^{2m} dx \le CC(n) \|a\|^2_{L^2} \le CC_n.
\end{aligned}
$$

This concludes the proof of Theorem 3.2. $\square$

3.3. Proof of Lemmas 3.3 and 3.4

Proof of Lemma 3.3. Let $j \le 0$. Let $T(u)$ be the degree s Taylor polynomial of $u \mapsto \psi(x-u)$, centered at the origin. We express $g_j(x)$ as follows:

$$
\begin{aligned}
g_j(x) = (\hat{a}_j)^\vee(x) = ((D^j_{A^*}\hat{a})^\vee * \psi)(x) &= \int_{\mathbb{R}^n} (D^j_{A^*}\hat{a})^\vee(u) \psi(x-u) du \\
&= \int_{\mathbb{R}^n} b^{-j} D_A^{-j} a(u) \psi(x-u) du = b^{-j} \int_{B_{j+1}} a(A^{-j}u) \left(\psi(x-u) - T(u)\right) du.
\end{aligned}
$$

Since $j \le 0$, with L being line between 0 and $u \in B_j$, the remainder term $R(u)$ satisfies the following estimates, with C_s a constant depending only on s:

$$
\begin{aligned}
|R(u)| \le C_s |u|^{s+1} \sup_{|\beta|=s+1} \sup_{z \in L} |\partial^\beta(\psi(x-z))|, \quad \text{with} \\
|\partial^\beta(\psi(x-z))| \le \frac{C_s}{(1+\rho(x-z))^r} \le \frac{C_s}{(1+\rho(x))^r}.
\end{aligned}
$$

By the vanishing moments condition on the atom, we continue the estimate (3.28):

$$\begin{aligned}|g_j(x)| &\leq \int_{B_{j+1}} |a(A^{-j}u)||R(u)|\frac{du}{b^j} \leq C\int_{B_{j+1}} |a(A^{-j}u)|\frac{|u|^{s+1}}{(1+\rho(x))^r}\frac{du}{b^j} \\ &\leq \frac{C}{(1+\rho(x))^r}\left(\int_{B_{j+1}} |a(A^{-j}u)|\cdot\rho(u)^{(s+1)\zeta_-}\frac{du}{b^j}\right) \\ &\leq C\frac{b^{j(s+1)\zeta_-}}{(1+\rho(x))^r}\int_{B_1}|a(y)|dy \leq Cb^{1/2}\frac{b^{j(s+1)\zeta_-}}{(1+\rho(x))^r}.\end{aligned}$$

This gives the first estimate. For the second estimate, we fix $j \geq 0$. For a general $x \in \mathbb{R}^n$, we apply Hölder's inequality to (3.28), which gives a constant C depending only on ψ and b such that

$$|g_j(x)| = \left|\frac{1}{b^j}\int_{B_{j+1}} a(A^{-j}u)\psi(x-u)du\right| \leq \frac{C}{b^j}\left(\int_{B_1}|a(y)|^2 b^j dy\right)^{1/2} \leq Cb^{-j/2}.$$

If $x \in B_{j+1}^c$, there exists $R > j+1 \geq 1$ such that $x \in B_{R+1}\backslash B_R$. Then if $y \in B_{j+1}$, we have

$$\rho(x-y) \geq b^{R-\omega-2}, \qquad \text{that is,} \qquad x-y \in B_{R-\omega-1}^c. \tag{3.28}$$

Returning to estimating g_j, we have

$$|g_j(x)| \leq b^{-j/2}\left(\int_{B_{j+1}}|\psi(x-u)|^2du\right)^{1/2}$$

With x, y as in (3.28), since $x-y \in B^c_{R-\omega-1}$, along with $\psi \in S$, and $r' = r+1/2 > 1/2$, we have a constant C, depending only on A, r, and r' such that

$$\int_{B_{j+1}} |\psi(x-y)|^2 dy \leq \int_{B^c_{R-\omega-1}} |\psi(u)|^2 du \leq C \int_{B^c_{R-\omega-1}} \frac{1}{\rho(u)^{2r'}} du = C \sum_{i=R-\omega}^{\infty} \int_{B_{i+1}\setminus B_i} b^{-2ir'} du$$
$$= C \sum_{i=R-\omega}^{\infty} b^{i(1-2r')} = C \frac{1}{1-b^{1-2r'}} b^{(1-2r')R} = C b^{R(1-2r')} = \frac{C}{\rho(x)^{2r'-1}},$$

the above sum converges. Then we have

$$|g_j(x)| \lesssim \frac{b^{-j/2}}{\rho(x)^{r'-1/2}} = \frac{b^{-j/2}}{\rho(x)^{r}},$$

completing the proof of lemma 3.3. □

Proof of Lemma 3.4. Fix such a Q. Without loss of generality, we assume $\|g\|_{K_1^{\frac{1}{p}-1,p}} = 1$. Then it suffices to prove the existence of a uniform constant C, independent of g and Q, such that

$$\sum_{k=j+2\omega}^{\infty} \left(\int_{\mathcal{A}_k} |g * Q(x)| dx \right)^p b^{k(1-p)} \leq C,$$

which will follow from the following pointwise estimate on $(g * Q)(x)$. Define $J(l) = \int_{\mathcal{A}_l} |g(y)| dy$. If $x \in \mathcal{A}_k$ and $k \geq j + 2\omega$, then we claim there exists $C = C(b, \omega, r)$ such that

$$|(g * Q)(x)| \leq C \left[b^{-r(k-2\omega)} \sum_{l=-\infty}^{a_1} J(l) + \sum_{l=a_2}^{\infty} J(l) b^{-r(l-\omega)} + \sum_{l=a_1+1}^{a_2-1} \left| \int_{\mathcal{A}_l} g(y) Q(x-y) dy \right| \right]. \tag{3.29}$$

Indeed, assuming (3.29) holds, we integrate $g * Q$ over the annulus $\mathcal{A}_k$:

$$\int_{\mathcal{A}_k} |(g*Q)(x)|dx \le C \int_{\mathcal{A}_k} \left(b^{-r(k-2\omega)} \sum_{l=-\infty}^{a_1} J(l) + \sum_{l=a_2}^{\infty} J(l) b^{-r(l-\omega)} + \sum_{l=a_1+1}^{a_2-1} \left| \int_{\mathcal{A}_l} g(y)Q(x-y)dy \right| \right) dx$$
$$\le C \left(b^{-r(k-2\omega)} b^k + \sum_{l=a_2}^{\infty} J(l) b^{-r(l-\omega)} b^k + \sum_{l=a_1+1}^{a_2-1} \int_{\mathcal{A}_k} \left(\int_{\mathcal{A}_l} |g(y)Q(x-y)|dy \right) dx \right).$$

We define the following:

$$\begin{aligned} P_L &= b^{-r(k-2\omega)} b^k, \\ P_H &= \sum_{l=a_2}^{\infty} J(l) b^{-r(l-\omega)} b^k \\ P_k &= \sum_{l=a_1+1}^{a_2-1} \int_{\mathcal{A}_k} \left(\int_{\mathcal{A}_l} |g(y)Q(x-y)|dy \right) dx. \end{aligned}$$

Summing over k, we have

$$\sum_{k=j+2\omega}^{\infty} \left(\int_{\mathcal{A}_k} |(g*Q)(x)|dx \right)^p b^{k(1-p)} \le C \sum_{k=j+2\omega}^{\infty} (P_L^p + P_H^p + P_k^p) b^{k(1-p)}.$$

To evaluate the sum $\sum_{k=j+2\omega}^{\infty} P_L^p b^{k(1-p)}$, we note that $r > \frac{1}{p}$ and $k \ge j + 2\omega \ge 0$. Then there is a constant C_L, depending only on ω, r, p, and b, such that

$$\sum_{k=2\omega}^{\infty} P_L^p b^{k(1-p)} = \sum_{k=j+2\omega}^{\infty} b^{pk-pr(k-2\omega)} b^{k(1-p)} = b^{2\omega rp} \sum_{k=j+2\omega}^{\infty} b^{k(1-rp)} \le C_L.$$

To evaluate the sum $\sum_{k=j+2\omega}^{\infty} P_H^p$, we note that if $r > \frac{1}{p} > 1$ and $l \ge a_2$, then $p-1-rp = p(1-r)-1 < 0$ is equivalent to $l(p-1-rp) \le a_2(p-1-rp)$. By the definition of

$a_2(k)$, there exists a constant c, depending only on b, r and ω, such that

$$\begin{aligned}
\sum_{k=j+2\omega}^{\infty} P_H^p b^{k(1-p)} &\leq \sum_{k=j+2\omega}^{\infty} \left(\sum_{l=a_2}^{\infty} J(l)^p b^{-rp(l-\omega)} b^{kp} \right) b^{k(1-p)} = c \sum_{k=j+2\omega}^{\infty} b^k \sum_{l=a_2}^{\infty} J(l)^p b^{l(1-p)} b^{l(p-1-rp)} \\
&\leq c \sum_{k=j+2\omega}^{\infty} b^k b^{a_2(p-1-rp)} \left(\sum_{l=a_2}^{\infty} J(l)^p b^{l(1-p)} \right) \\
&\leq c \sum_{k=j+2\omega}^{\infty} b^k b^{a_2(p-1-rp)} \|g\|^p_{K_1^{\frac{1}{p}-1,p}} \leq c \sum_{k=j+2\omega}^{\infty} b^k b^{a_2(p-1-rp)} \\
&\leq \max \left\{ \sum_{k=j+2\omega}^{\infty} b^k b^{(k+\log(1+b^{\omega}))(p-1-rp)}, \sum_{k=j+2\omega}^{\infty} b^k b^{(k+2\omega)(p-1-rp)} \right\},
\end{aligned}$$

Now in the first estimate, there is a constant C, depending only on b, ω, r and p such that the power of b is given by

$$k + k(p-1-rp) + \log(1+b^{\omega})(p-1-rp) = kp(1-r) + C.$$

In the second estimate, the power of b is given by

$$k + k(p-1-rp) + 2\omega(p-1-rp) = kp(1-r) + C.$$

Since $r > \frac{1}{p} > 1$, both estimates lead to a uniform bound for $\sum_{k=j+2\omega}^{\infty} P_H^p b^{k(1-p)}$.

We now estimate the sum $\sum_{k=j+2\omega}^{\infty} P_k^p b^{k(1-p)}$. By Fubini and the fact that $a_2 - a_1$ depend only on ω, b, and the estimate in the lemma is proved.We have a constant C, depending only on ω and b, such that

$$\sum_{k=j+2\omega}^{\infty} P_k^p b^{k(1-p)} = \sum_{k=j+2\omega}^{\infty} \left(\sum_{l=a_1+1}^{a_2-1} \int_{\mathcal{A}_k} \int_{\mathcal{A}_l} |g(y)Q(x-y)| dy dx \right)^p b^{k(1-p)}$$
$$\leq \sum_{k=j+2\omega}^{\infty} \left(\sum_{l=a_1+1}^{a_2-1} \left(\int_{\mathcal{A}_l} |g(y)| dy \right) \|Q\|_{L^1} \right)^p b^{k(1-p)}$$
$$\leq \sum_{k=j+2\omega}^{\infty} \left(\sum_{l=a_1+1}^{a_2-1} J(l)^p \right) b^{k(1-p)} \leq C(\omega, b) \sum_{k=j+2\omega}^{\infty} J(l)^p b^{k(1-p)} \leq C(\omega, b).$$

It remains to prove the pointwise estimate (3.29). We start with a simple decomposition. With $a_1(k)$ and $a_2(k)$ as before, we have

$$(g * Q)(x) = \sum_{l=-\infty}^{\infty} \int_{\mathcal{A}_l} g(y)Q(x-y)dy = \sum_{l=-\infty}^{a_1} + \sum_{a_1+1}^{a_2-1} + \sum_{l=a_2}^{\infty} \int_{\mathcal{A}_l} g(y)Q(x-y)dy,$$

and we denote the first two sums by S_L and S_H, respectively. To estimate S_L, let $l \leq a_1$ and let $y \in \mathcal{A}_l$. We claim that $\rho(x-y) \geq b^j$. To prove this, we use the anisotropic triangle inequality:

$$\rho(x-y) \geq b^{-\omega}\rho(x) - \rho(y) = b^{-\omega} b^k - b^l \geq b^{-\omega} b^k - b^{a_1}$$
$$\geq b^k (b^{-\omega} - b^{a_1 - k}) \geq b^k b^{-\omega-1} \geq b^{k-2j} \geq b^j.$$

Then by our assumption on Q, we have

$$|Q(x-y)| \leq \frac{1}{\rho(x-y)^r} \leq \frac{1}{b^{r(k-2\omega)}}.$$

Returning to S_L, we have

$$\begin{aligned}
|S_L| &= \left| \sum_{l=-\infty}^{a_1} \int_{\mathcal{A}_l} g(y)Q(x-y)dy \right| \leq \sum_{l=-\infty}^{a_1} \int_{\mathcal{A}_l} |Q(x-y)| \cdot |g(y)| dy \\
&\leq \left(\sum_{l=-\infty}^{a_1} \int_{\mathcal{A}_l} |g(y)| dy \right) b^{-r(k-2\omega)} = b^{-r(k-2\omega)} \sum_{l=-\infty}^{a_1} J(l).
\end{aligned}$$

To estimate S_H, we let $x \in \mathcal{A}_k$, let $l \geq a_2$ and let $y \in \mathcal{A}_l$. We claim that $\rho(x-y) \geq b^j$. Indeed, starting with the anisotropic triangle inequality, we have

$$\begin{aligned}
\rho(x-y) &\geq b^{-\omega}\rho(y) - \rho(x) = b^{-\omega}b^l - b^k \geq b^{-\omega}b^{a_2} - b^k \\
&= b^k(b^{a_2-k-\omega} - 1) \geq b^k b^{-\omega} = b^{k-\omega} \geq b^{j+2\omega-\omega} \geq b^j.
\end{aligned}$$

The condition on Q gives the estimate $Q(x-y) \leq \frac{1}{\rho(x-y)^r}$. Since $a_2 \geq k+2\omega$. Then there exists c, depending only on A, such that

$$\rho(x-y) \geq b^{-\omega}\rho(y) - \rho(x) = b^{-\omega}b^l - b^k \geq b^{-\omega}b^l - b^{a_2-2\omega} \geq b^{-\omega}b^l - b^{l-2\omega},$$

so $\dfrac{1}{\rho(x-y)} \leq \dfrac{c}{b^{\omega-l}}$. We now complete our estimate on S_H:

$$\begin{aligned}
|S_H| &= \left| \sum_{l=a_2}^{\infty} \int_{\mathcal{A}_l} g(y)Q(x-y)dy \right| \leq \sum_{l=a_2}^{\infty} \int_{\mathcal{A}_l} \frac{|g(y)|}{\rho(x-y)^r} dy \\
&\leq C \sum_{l=a_2}^{\infty} \int_{\mathcal{A}_l} \frac{|g(y)|}{b^{r(\omega-l)}} dy = \sum_{l=a_2}^{\infty} b^{-r(l-\omega)} J(l).
\end{aligned}$$

This completes the proof of (3.29). $\square$

CHAPTER IV

FOURIER TRANSFORM OF H^P_A

4.1. Introduction

We now look at a well-known problem of the characterization of $\hat{f}$ for $f \in H^p$. Coifman [Coi74a] characterized all such $\hat{f}$ on $\mathbb{R}$ using entire functions of exponential type. In higher dimensions necessary conditions have been studied by a number of authors [Col82, GCK01, Tai66]. In particular, Taibleson and Weiss [GCK01] showed that for $p \in (0, 1]$, the Fourier transform of $f \in H^p(\mathbb{R}^n)$ is continuous and satisfies the following estimate:

$$|\hat{f}(\xi)| \leq C\|f\|_{H^p}|\xi|^{n(\frac{1}{p}-1)}. \tag{4.1}$$

This leads to the following consequences; see [GCRdF85, III.7], [Tai66] for more details. At the origin, the estimate (4.1) forces $f \in H^p \cap L^1$ to have vanishing moments, as seen by the degree of 0 of $\hat{f}$ at the origin, illustrating the necessity of the vanishing moments of the atoms. Away from the origin, the polynomial growth is sharp, as given by an extension of the Hardy-Littlewood inequality for $f \in H^p$, $0 < p \leq 1$,

$$\int_{\mathbb{R}^n} |\xi|^{n(p-2)}|\hat{f}(\xi)|^p d\xi \leq C\|f\|^p_{H^p}. \tag{4.2}$$

The estimate (4.1) also sheds light on multiplier operators of H^p. When paired with the molecular characterization of H^p, it shows that the multiplier operator T_m : $H^p \to H^p$ is bounded provided the multiplier m satisfies the (integral) Hörmander condition. On the other hand, if T_m is any bounded multiplier operator on H^p, then m is necessarily continuous and bounded on $\mathbb{R}^n \backslash \{0\}$. The main purpose of this chapter is to extend (4.1) from the isotropic (classical) setting to anisotropic Hardy spaces H_A^p associated with a dilation matrix A. We recall that the adjoint A^* is a dilation matrix, with its associated quasinorm ρ_*.

Theorem 4.1. *Let $p \in (0,1]$. If $f \in H_A^p(\mathbb{R}^n)$, then $\hat{f}$ is a continuous function and satisfies*

$$|\hat{f}(\xi)| \leq C\|f\|_{H_A^p}\ \rho_*(\xi)^{\frac{1}{p}-1} \tag{4.3}$$

with $C = C(A,p)$.

Theorem 4.1 leads to similar consequences as in the isotropic setting. At the origin, we obtain a sharper order for the convergence of $\hat{f}(\xi)$ as $\xi \to 0$. This is given by Corollary 4.2, and shows the necessity of vanishing moments for anisotropic atoms in H_A^p. We then obtain necessary conditions for a function m to be a multiplier on H_A^p, given by Corollary 4.3. Lastly, we show in Corollary 4.4 that the function $|\hat{f}(\xi)|^p \rho_*(\xi)^{p-2}$ is integrable, which is a generalization of Hardy-Littlewood's inequality (4.2). In Theorem 4.5, we further improve this estimate using rearrangement functions as in the work of García-Cuerva and Kolyada [Tai66], though we use a slightly different argument.

Lastly, the notion of an atom can be generalized to a molecule (See Theorem 2.3): A function f is a molecule localized around $x_0 + B_k$ if it satisfies the above vanishing moment condition, and

$$\left(\frac{1}{|B_k|}\int_{x_0+B_k}|f(x)|^q dx\right)^{1/q} \leq C|B_k|^{-1/p},$$
$$|f(x)| \leq C|B_k|^{-1/p}\rho(A^{-k}(x-x_0))^{-\delta} \text{ for } x \in x_0 + B_k^c.$$

All such molecules are in H_A^p. In particular, f does not have to be compactly supported, and any Schwartz function satisfying the vanishing moments is immediately such a molecule.

4.2. Proof of Theorem 4.1

To prepare for the following two lemmas, we recall two basic facts. If we define the dilation operator by $D_A(f)(x) = f(Ax)$, then (2.9) states it commutes with the Fourier transform by the following identity for all $j \in \mathbb{Z}$:

$$b^j(D^j_{A^*}\mathcal{F}D^j_A f)(\xi) = \hat{f}(\xi).$$

Second, the eccentricities of A^* are the same as A, that is, (1.11) and (1.12) hold with the same constants c_A, ζ_+, ζ_-. Indeed, A^* has the same eigenvalues as A.

Lemma 4.1. *Let a be a (p,q,s) atom supported on $x_0 + B_k$ for some $x_0 \in \mathbb{R}^n$ and $k \in \mathbb{Z}$. Suppose α is a multi-index, with $|\alpha| \leq s$. There exists a constant $C = C(s)$ such that*

$$|\partial^\alpha(\mathcal{F}D^k_A a)(\xi)| \leq Cb^{-\frac{k}{q}}\|a\|_q \min\{1, |\xi|^{s-|\alpha|+1}\}. \tag{4.4}$$

Proof. Without loss of generality, we can assume a is supported on B_k, so $\mathrm{supp}(D_A^k a) \subset B_0$. Fixing a multi-index $|\alpha| \le s$, we have

$$|\partial^\alpha(\mathcal{F}D_A^k a)(\xi)| = \left| \int_{B_0} (-2\pi i x)^\alpha (D_A^k a)(x) e^{-2\pi i \langle x,\xi\rangle} dx \right|.$$

Let $T(x)$ be the degree $s-|\alpha|$ Taylor polynomial of the function $x \mapsto e^{-2\pi i\langle x,\xi\rangle}$ centered at the origin. Using the vanishing moments of an atom, we have

$$\begin{aligned}
&|\partial^\alpha(\mathcal{F}D_A^k a)(\xi)| = \left| \int_{B_0} (-2\pi i x)^\alpha (D_A^k a)(x) e^{-2\pi i \langle x,\xi\rangle} dx \right| \\
&= \left| \int_{B_0} (-2\pi i x)^\alpha (D_A^k a)(x) \left[e^{-2\pi i \langle x,\xi\rangle} - T(x) \right] dx \right| \le C \int_{B_0} |x^\alpha| |a(A^k x)| |x|^{s-|\alpha|+1} |\xi|^{s-|\alpha|+1} dx \\
&\le C|\xi|^{s-|\alpha|+1} \int_{B_0} |x|^{s+1} |a(A^k x)| dx \le C|\xi|^{s-|\alpha|+1} \int_{B_k} |a(y)| \frac{dy}{b^k} \le C|\xi|^{s-|\alpha|+1} b^{-k/q} \|a\|_q.
\end{aligned}$$

The third line is a consequence of Taylor's remainder formula. To obtain the other estimate, we estimate without the Taylor approximation

$$\begin{aligned}
|\partial^\alpha(\mathcal{F}D_A^k a)(\xi)| &= \left| \int (-2\pi i x)^\alpha (D_A^k a)(x) e^{-2\pi i \langle x,\xi\rangle} dx \right| \le C \int_{B_0} |x|^{|\alpha|} |a(A^k x)| dx \\
&\le C\ b^{-k} \int_{B_k} |a(y)| dy \le C b^{-k/q} \|a\|_q.
\end{aligned}$$

This completes the proof. $\square$

Lemma 4.2. *Let a be a (p,q,s) atom supported on $x_0 + B_k$ for some $x_0 \in \mathbb{R}^n$ and $k \in \mathbb{Z}$. Then we have the following bound, with C independent of a,*

$$|\hat{a}(\xi)| \le C\rho_*(\xi)^{\frac{1}{p}-1}. \tag{4.5}$$

Proof. Setting $\alpha = 0$, (4.4) reduces to the following estimate

$$|\hat{a}(\xi)| \leq \begin{cases} Cb^{k(1-1/p)}b^{(s+1)k\zeta_-}\rho_*(\xi)^{(s+1)\zeta_-} & \text{for } \rho_*(\xi) \leq b^{-k}, \\ Cb^{k(1-1/p)} & \text{for all } \xi. \end{cases} \tag{4.6}$$

Indeed, with (2.9) and setting $\alpha = 0$ in (4.4),

$$\begin{aligned} |\hat{a}(\xi)| &= |b^k(\mathcal{F}D_A^k a)(A^{*k}\xi))| \leq Cb^k b^{-\frac{k}{q}}\|a\|_q \min(1, |A^{*k}\xi|^{s+1}) \\ &\leq Cb^{k(1-1/p)} \min(1, |A^{*k}\xi|^{s+1}). \end{aligned}$$

This immediately yields the second estimate (4.6). To see the first estimate, we take $\rho_*(\xi) \leq b^{-k}$, which is equivalent to $A^{*k}\xi \in B_1^*$. Hence, by (1.11), $|(A^*)^k\xi)| \leq c_A b^{k\zeta_-}\rho_*(\xi)^{\zeta_-}$. Thus,

$$|\hat{a}(\xi)| \leq Cb^{k(1-1/p)}(b^{k\zeta_-}\rho_*(\xi)^{\zeta_-})^{s+1}.$$

This shows (4.6), which we will use to prove (4.5).

If $\rho_*(\xi) \leq b^{-k}$, then

$$\begin{aligned} |\hat{a}(\xi)| &\leq Cb^{k((1-1/p)+(s+1)\zeta_-)}\rho_*(\xi)^{(s+1)\zeta_-} \\ &\leq C\rho_*(\xi)^{-(1-1/p)-(s+1)\zeta_-}\rho_*(\xi)^{(s+1)\zeta_-} = C\rho_*(\xi)^{\frac{1}{p}-1}. \end{aligned}$$

In the second inequality we used the fact that $1 - \frac{1}{p} + (s+1)\zeta_- \geq 0$, since (p, q, s) is admissible. If $\rho_*(\xi) > b^{-k}$, then by (4.6), we have

$$|\hat{a}(\xi)| \leq Cb^{-k(1/p-1)} \leq C\rho_*(\xi)^{\frac{1}{p}-1},$$

where the last inequality holds since $1/p - 1 \geq 0$. This completes the proof of the lemma. $\square$

We are now ready to prove Theorem 4.1 by extending (4.5) to every $f \in H^p_A$.

Proof of Theorem 4.1. Let $f \in H^p_A$. By the atomic decomposition of H^p_A, we can find coefficients (λ_i) and atoms (a_i) such that $f = \sum \lambda_i a_i$ (in H^p_A-norm) and $2\|f\|_{H^p_A} \geq \|(\lambda_i)\|_{\ell^p}$. This sum converges in H^p_A-norm, which implies convergence in $\mathcal{S}'$. So by taking the Fourier transform on f, we have $\hat{f} = \sum_i \lambda_i \hat{a}_i$, converging in $\mathcal{S}'$. By (4.5) and the fact that $(\lambda_i) \in \ell^1$,

$$\sum_{i=1}^{\infty} |\lambda_i||\hat{a}_i(\xi)| \leq C \sum_{i=1}^{\infty} |\lambda_i| \ \rho_*(\xi)^{\frac{1}{p}-1} \leq 2C\rho_*(\xi)^{\frac{1}{p}-1}\|f\|_{H^p} < \infty.$$

Therefore, the sum $\hat{f}(\xi) = \sum_i \lambda_i \hat{a}_i(\xi)$ converges absolutely on $\mathbb{R}^n$. Furthermore, on each compact set K, $\rho_*(\xi)$ is bounded by a constant C' independent of a, so the absolute convergence above is also uniform on each compact set K. With $\hat{a}_i$ infinitely differentiable (hence continuous) for all i, we conclude $\hat{f}(\xi)$ is continuous on all compact sets K, and hence on $\mathbb{R}^n$. $\square$

4.3. Applications of Theorem 4.1

We now consider consequences of Theorem 4.1. The first corollary refines the order of 0 at the origin, and the second gives necessary conditions on a multiplier m on H^p_A. The third corollary is the Hardy-Littlewood inequality on Hardy spaces, which will be strengthened by a rearrangement argument.

Corollary 4.2. *Let $f \in H^p_A(\mathbb{R}^n)$, $0 < p \le 1$. Then,*

$$\lim_{\xi \to 0} \frac{\hat{f}(\xi)}{\rho_*(\xi)^{\frac{1}{p}-1}} = 0. \tag{4.7}$$

Proof. We start by verifying this on an atom a, with support B_k. By (4.6), if $\rho_*(\xi) \le b^{-k}$, we have

$$|\hat{a}(\xi)| \le C b^{k(1-1/p)} b^{(s+1)k\zeta_-} \rho_*(\xi)^{(s+1)\zeta_-}.$$

Since $s \ge \lfloor (1/p - 1)\zeta_- \rfloor$, this implies $(s+1)\zeta_- > \frac{1}{p} - 1$. Therefore, we obtain (4.7) for atoms;

$$\lim_{\xi \to 0} \frac{\hat{a}(\xi)}{\rho_*(\xi)^{\frac{1}{p}-1}} = 0.$$

Now if $f \in H^p_A$, we can decompose $f = \sum_i \lambda_i a_i$, for $(\lambda_i) \in \ell^p$ and (p, q, s)-atoms a_i. Thus,

$$\frac{|\hat{f}(\xi)|}{\rho_*(\xi)^{\frac{1}{p}-1}} \le \sum_{i=1}^{\infty} \frac{|\hat{a}_i(\xi)|}{\rho_*(\xi)^{\frac{1}{p}-1}} |\lambda_i|.$$

By (4.5) and the fact that $(\lambda_i) \in \ell^1$, we can apply the Dominated Convergence Theorem to the above sum (treated as an integral). Since each term in the sum goes to 0 as $\xi \to 0$ we obtain (4.7). □

Corollary 4.3, which is a generalization of [GCRdF85, Theorem III.7.31], gives a necessary condition for multipliers on anisotropic Hardy spaces H^p_A.

Corollary 4.3. *Suppose m is a multiplier on H^p_A, $0 < p \le 1$. That is, the following operator is bounded:*

$$T_m : H^p_A \to H^p_A, \qquad T_m(f) = (m\hat{f})^\vee,$$

with $M > 0$ as the operator norm of T_m. Then, m is continuous on $\mathbb{R}^n\backslash\{0\}$ and uniformly bounded with $\|m\|_\infty \leq CM$.

Proof. Fix $0 < p \leq 1$. For $k \in \mathbb{Z}$, we denote $f_k(x) = b^{k/p} f(A^k x)$. Then, this dilation is invariant under H^p_A (and L^p) norm: $\|f_k\|_{H^p_A} = \|f\|_{H^p_A}$. Under the Fourier transform, we have

$$\hat{f}_k(\xi) = b^{k(\frac{1}{p}-1)} \hat{f}((A^*)^{-k}\xi).$$

Then by (4.3), the following estimate holds for all $k \in \mathbb{Z}, \xi \in \mathbb{R}^n$,

$$|m(\xi)\hat{f}((A^*)^{-k}\xi)| \leq CM\|f\|_{H^p}\ \rho_*(\xi)^{\frac{1}{p}-1} b^{k(1-\frac{1}{p})}.$$

If $\xi \in B^*_{k+1}\backslash B^*_k$, then $(A^*)^{-k}\xi \in B^*_1\backslash B^*_0$, and we have

$$|m(\xi)\hat{f}((A^*)^{-k}\xi)| \leq CM\|f\|_{H^p}.$$

This estimate will force m to be bounded if there exists $f \in H^p_A$ such that $\hat{f}$ does not vanish on the unit annulus $B^*_1\backslash B^*_0$. Take $g \in C^\infty_c$, supported on $B^*_2\backslash B^*_{-1}$ such that g is identically 1 on $B^*_1\backslash B^*_0$. Setting $\hat{f} = g$, f is immediately in the Schwartz class $\mathcal{S}$, with vanishing moments of all order. In particular, f is a molecule for H^p_A (see Remark in [Bow03, Section 9]), hence $f \in H^p_A$. This shows that $||m||_\infty \leq CM$. Moreover, by Theorem 4.1 the function $\xi \mapsto m(\xi)\hat{f}((A^*)^{-k}\xi)$ is continuous for each $k \in \mathbb{Z}$. Thus, m is continuous on $\mathbb{R}^n\backslash\{0\}$. □

Corollary 4.4. *If $f \in H^p_A(\mathbb{R}^n)$, $0 < p \leq 1$, then*

$$\int_{\mathbb{R}^n} |\hat{f}(\xi)|^p \rho_*(\xi)^{p-2} d\xi \leq C\|f\|^p_{H^p_A}. \tag{4.8}$$

Proof. Suppose a $(p,2,s)$ atom a is supported on x_0+B_k. We claim that

$$\int_{\mathbb{R}^n} |\hat{a}(\xi)|^p \rho_*(\xi)^{p-2} d\xi \le C. \tag{4.9}$$

Indeed, by (4.6) we can estimate the integral on B^*_{-k}

$$\int_{B^*_{-k}} |\hat{a}(\xi)|^p \rho_*(\xi)^{p-2} d\xi \le C^p b^{k(p-1)} b^{p(s+1)k\zeta_-} \int_{B^*_{-k}} \rho_*(\xi)^{p-2+p(s+1)\zeta_-} d\xi \le C^p.$$

For the integral outside of B^*_{-k}, we use Hölder's inequality

$$\begin{aligned}\int_{(B^*_{-k})^c} |\hat{a}(\xi)|^p \rho_*(\xi)^{p-2} d\xi &\le C \left(\int_{(B^*_{-k})^c} |\hat{a}(\xi)|^2 d\xi\right)^{\frac{p}{2}} \left(\int_{(B^*_{-k})^c} \rho_*(\xi)^{-2} d\xi\right)^{\frac{2-p}{2}} \\ &\le C||a||_2^p b^{-k(\frac{p}{2}-1)} \le C.\end{aligned}$$

Combining these two estimates, we obtain (4.9). Now let $f \in H^p_A$ have an atomic decomposition $f = \sum_i \lambda_i a_i$ with $\|(\lambda_i)\|_{\ell^p} \le 2\|f\|_{H^p_A}$. Since $p \in (0,1]$, we have

$$\int_{\mathbb{R}^n} |\hat{f}(\xi)|^p \rho_*(\xi)^{p-2} d\xi \le \sum_i |\lambda_i|^p \int_{\mathbb{R}^n} |\hat{a}_i(\xi)|^p \rho_*(\xi)^{p-2}\ d\xi \le C \sum_i |\lambda_i|^p \le C\|f\|^p_{H^p_A}.$$

This shows (4.8). $\square$

The following result improves (4.8) by extending [GCRdF85, Lemma 3.1] to the anisotropic setting. We denote $S_0(\mathbb{R}^n)$ as the collection of all measurable functions f, finite almost everywhere, whose distributional functions satisfy

$$d_f(t) = |\{x \in \mathbb{R}^n : |f(x)| > t\}| < \infty \qquad \text{for all } t > 0. \tag{4.10}$$

For $f \in S_0(\mathbb{R}^n)$, its rearrangement function is defined by

$$f^\star(t) = \inf\{s > 0 : d_f(s) \le t\}.$$

We recall the following facts regarding the rearrangement function. If $f \le g$ on $\mathbb{R}^n$, then $f^\star(t) \le g^\star(t)$ for all t. For all $\lambda > 0$,

$$(|f|^\lambda)^\star(t) = f^\star(t)^\lambda. \tag{4.11}$$

These follow immediately from the definition. Lastly,

$$\int_0^t \left(\sum_j f_j\right)^\star (u)du \le \sum_j \int_0^t f_j^\star(u)du, \tag{4.12}$$

for all $t > 0$, provided the right-hand side is finite; see [BS88, Chapter 2, §3].

Theorem 4.5. *Let $\epsilon > 0, 0 < p < 1$ and define $\lambda = \frac{1}{p} - 1 + \epsilon$. Then, there exists C such that for all $f \in H_A^p(\mathbb{R}^n)$,*

$$\left(\int_0^\infty t^{\epsilon p - 1} F_\epsilon^\star(t)^p dt\right)^{1/p} \le C\|f\|_{H_A^p}, \tag{4.13}$$

with $F_\epsilon(\xi) = \rho_*(\xi)^{-\lambda}|\hat{f}(\xi)|$.

To see why Theorem 4.5 strengthens (4.8), we observe that if $g(\xi) = 1/\rho_*(\xi)$, a simple computation shows

$$g^\star(t) \simeq 1/t. \tag{4.14}$$

If $f, g \in S_0(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |f(\xi)g(\xi)|dx \leq \int_0^\infty f^\star(t)g^\star(t)dt.$$

Together, these two facts can be used to show the left-hand side of (4.13) majorizes the left-hand side of (4.8).

Proof of Theorem 4.5. We will prove the following estimate for all $f \in H_A^p$:

$$\left(\int_0^\infty t^{\epsilon p-2}\left[\int_0^t F_\epsilon^\star(u)^p du\right] dt\right)^{1/p} \leq C\|f\|_{H_A^p}, \tag{4.15}$$

which implies (4.13). Indeed, the rearrangement function is always decreasing for $0 < t < \infty$. Thus, $F_\epsilon^\star(t)^p \leq \frac{1}{t}\int_0^t F_\epsilon^\star(u)^p du$. Then,

$$\int_0^\infty t^{\epsilon p-1}F_\epsilon^\star(t)^p dt \leq \int_0^\infty t^{\epsilon p-1}\left(\frac{1}{t}\int_0^t F_\epsilon^\star(u)^p du\right) dt = \int_0^\infty t^{\epsilon p-2}\left(\int_0^t F_\epsilon^\star(u)^p du\right) dt.$$

We first prove (4.15) for unit atoms. Using a dilation argument, we extend it to all atoms, and to any $f \in H_A^p$ using the atomic decomposition.

Let f be a unit $(p, 2, s)$ atom, that is, an atom supported on $x_0 + B_0$. Without loss of generality, we set $x_0 = 0$. On unit atoms, the estimates (4.5) and (4.6) reduce to

$$\|\hat{f}\|_\infty \leq \begin{cases} \rho_*(\xi)^{(s+1)\zeta_-} & \text{for } \xi \in B_0^* \\ \rho_*(\xi)^{\frac{1}{p}-1} & \text{for all } \xi. \end{cases}$$

This implies

$$F_\epsilon(\xi) \le \begin{cases} \rho_*(\xi)^{\zeta_-(s+1)-\lambda} & \text{for } \xi \in B_0^*, \\ \rho_*(\xi)^{\frac{1}{p}-1-\lambda} & \text{for all } \xi, \end{cases}$$

where the first estimate has a positive power, and the second has a negative power. These give $\|F_\epsilon\|_\infty \le C$, and $F_\epsilon(\xi) \le C\rho_*(\xi)^{-\lambda}$, which by the properties of the rearrangement function and (4.14), imply

$$F_\epsilon^\star(t) \le C \min\{1, t^{-\lambda}\}.$$

With these estimates,

$$\int_0^\infty t^{\epsilon p-2}\left(\int_0^t F_\epsilon^\star(u)^p du\right) dt = \int_0^1 + \int_1^\infty t^{\epsilon p-2}\left(\int_0^t F_\epsilon^\star(u)^p du\right) dt = I_1 + I_2.$$

By the fact that $F_\epsilon^\star(t) \le C$, we have $I_1 \le C$. To estimate I_2,

$$\begin{aligned} I_2 &\le \int_1^\infty t^{\epsilon p-2}\left(\int_0^t F_\epsilon^\star(u)^p du\right) dt \le \int_1^\infty t^{\epsilon p-2}\left(\int_0^t u^{-\lambda p} du\right) dt \\ &\simeq \int_1^\infty t^{\epsilon p-2} t^{1-\lambda p} dt = \int_1^\infty t^{p-2} dt \le C. \end{aligned}$$

Since $\|f\|_{H_A^p} \le C$ for all atoms, we have (4.15) for unit atoms.

We now extend it to all atoms using a dilation argument. Let f be a general $(p, 2, s)$ atom supported on B_k. Then the dilated atom $f_k(x) = b^{k/p} f(A^k x)$ is an atom with the same H_A^p-norm, but supported on B_0, that is, f_k is a unit atom. Denoting

$G_\epsilon(\xi) = \rho_*(\xi)^{-\lambda}\ |\widehat{f}_k(\xi)|$, we have just shown that

$$\int_0^\infty t^{\epsilon p-2}\left(\int_0^t G_\epsilon^\star(u)^p du\right) dt \le C.$$

The fact that (4.15) holds for all atoms follows if we can show that the above quantity is the same if we replace G_ϵ by $F_\epsilon(\xi) = \rho_*(\xi)^{-\lambda}|\hat{f}(\xi)|$.

As before, we denote $D_{A^*}g(x) = g(A^*x)$. Then

$$G_\epsilon(\xi) = b^{-\epsilon k}(D_{A^*}^k F_\epsilon)(\xi).$$

The distribution function is affected as follows.

$$\begin{aligned} d_{G_\epsilon}(s) &= |\{\xi : G_\epsilon(\xi) > s\}| = |\{\xi : (D_{A^*}^{-k} F_\epsilon)(\xi) > sb^{\epsilon k}\}| \\ &= |\{\xi : F_\epsilon((A^*)^{-k}\xi) > sb^{\epsilon k}\}| = b^k|\{u : F_\epsilon(u) > sb^{\epsilon k}\}| = b^k d_{F_\epsilon}(sb^{\epsilon k}). \end{aligned}$$

This affects the rearrangement function as follows:

$$\begin{aligned} G_\epsilon^\star(t) &= \inf\{s : d_{G_\epsilon}(s) \le t\} = \inf\{s : d_{F_\epsilon}(sb^{\epsilon k}) \le b^{-k}t\} \\ &= b^{-\epsilon k}\inf\{r : d_{F_\epsilon}(r) \le b^{-k}t\} = b^{-\epsilon k}F_\epsilon^\star(b^{-k}t). \end{aligned} \tag{4.16}$$

By two changes of variables and (4.16), we have

$$\begin{aligned} \int_0^\infty t^{\epsilon p-2}\left(\int_0^t F_\epsilon^\star(u)^p du\right) dt &= \int_0^\infty t^{\epsilon p-2}\left(\int_0^t b^{p\epsilon k}G_\epsilon^\star(b^k u)^p du\right) dt \\ &= \int_0^\infty s^{\epsilon p-2}\left(\int_0^s G_\epsilon^\star(r)^p dr\right) ds \le C. \end{aligned}$$

This extends (4.15) to all atoms, and we now extend it to all $f \in H^p_A$. If $f \in H^p_A$, then we have the atomic decomposition

$$f = \sum_j \lambda_j a_j,$$

with $(p, 2, s)$ atoms a_j and $(\lambda_j) \in \ell^p$. Taking the Fourier transform, we have the following sum in the distributional and pointwise sense:

$$\hat{f}(\xi) = \sum_j \lambda_j \widehat{a_j}(\xi).$$

With $F_\epsilon(\xi) = |\xi|^{-\lambda}|\hat{f}(\xi)|$ and $p \in (0,1)$,

$$F_\epsilon(\xi)^p = \left(|\xi|^{-\lambda} \Big| \sum_j \lambda_j \widehat{a_j}(\xi) \Big| \right)^p \leq \sum_j |\lambda_j|^p \cdot \left(|\xi|^{-\lambda} |\widehat{a_j}(\xi)| \right)^p = \sum_j |\lambda_j|^p A_j(\xi)^p,$$

where $A_j(\xi) = |\xi|^{-\lambda}|\widehat{a_j}(\xi)|$. Recall that the rearrangement operation is order-preserving ($f \leq g \Rightarrow f^\star \leq g^\star$). By (4.11) and (4.12), we have

$$\int_0^t F_\epsilon^\star(u)^p du \leq \int_0^t \left(\sum_j |\lambda_j|^p A_j^p(\cdot) \right)^\star (u) du \leq \sum_j |\lambda_j|^p \int_0^t A_j^\star(u)^p du.$$

Therefore,

$$\begin{aligned} \int_0^\infty t^{\epsilon p - 2} \left[\int_0^t F_\epsilon^\star(u)^p du \right] dt &\leq \int_0^\infty t^{\epsilon p - 2} \left[\sum_j |\lambda_j|^p \int_0^t A_j^\star(u)^p du \right] dt \\ &= \sum_j |\lambda_j|^p \int_0^\infty t^{\epsilon p - 2} \left(\int_0^t A_j^\star(u)^p du \right) dt \leq C \sum_j |\lambda_j|^p, \end{aligned}$$

where the last inequality comes from (4.15) for all atoms. Taking the infimum over all possible atomic decompositions, we obtain (4.15) for all $f \in H^p_A$. □

REFERENCES CITED

[BB10] Á. Bényi and M. Bownik, *Anisotropic classes of homogeneous pseudodifferential symbols*, Studia Math. **200** (2010), no. 1, 41–66. MR 2720206 (2012c:47131)

[BLYZ08] M. Bownik, B. Li, D. Yang, and Y. Zhou, *Weighted anisotropic Hardy spaces and their applications in boundedness of sublinear operators*, Indiana Univ. Math. J. **57** (2008), no. 7, 3065–3100. MR 2492226 (2010a:42078)

[Bow03] M. Bownik, *Anisotropic Hardy spaces and wavelets*, Mem. Amer. Math. Soc. **164** (2003), no. 781, vi+122. MR 1982689 (2004e:42023)

[Bow05] ______, *Boundedness of operators on Hardy spaces via atomic decompositions*, Proc. Amer. Math. Soc. **133** (2005), no. 12, 3535–3542 (electronic). MR 2163588 (2006d:42028)

[BS85] A. Baernstein, II and E. Sawyer, *Embedding and multiplier theorems for* $H^P(\mathbf{R}^n)$, Mem. Amer. Math. Soc. **53** (1985), no. 318, iv+82. MR 776176 (86g:42036)

[BS88] C. Bennett and R. Sharpley, *Interpolation of operators*, Pure and Applied Mathematics, vol. 129, Academic Press Inc., Boston, MA, 1988. MR MR928802 (89e:46001)

[Coi74a] R. Coifman, *Characterization of Fourier transforms of Hardy spaces*, Proc. Nat. Acad. Sci. U.S.A. **71** (1974), 4133–4134. MR 0394000 (52 #14807)

[Coi74b] ______, *A real variable characterization of* H^p, Studia Math. **51** (1974), 269–274. MR 0358318 (50 #10784)

[Col82] L. Colzani, *Fourier transform of distributions in Hardy spaces*, Boll. Un. Mat. Ital. A (6) **1** (1982), no. 3, 403–410. MR 678657 (84c:46042)

[CT75] A. Calderón and A. Torchinsky, *Parabolic maximal functions associated with a distribution*, Advances in Math. **16** (1975), 1–64. MR 0417687 (54 #5736)

[CT77] ______, *Parabolic maximal functions associated with a distribution. II*, Advances in Math. **24** (1977), no. 2, 101–171. MR 0450888 (56 #9180)

[CW77] R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. **83** (1977), no. 4, 569–645. MR 0447954 (56 #6264)

[DL06] Y. Ding and S. Lan, *Some multiplier theorems for anisotropic Hardy spaces*, Anal. Theory Appl. **22** (2006), no. 4, 339–352. MR 2316761 (2008b:42020)

[FJ90] M. Frazier and B. Jawerth, *A discrete transform and decompositions of distribution spaces*, J. Funct. Anal. **93** (1990), no. 1, 34–170. MR 1070037 (92a:46042)

[FS72] C. Fefferman and E. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), no. 3-4, 137–193. MR 0447953 (56 #6263)

[FS82] G. Folland and E. Stein, *Hardy spaces on homogeneous groups*, Mathematical Notes, vol. 28, Princeton University Press, Princeton, N.J., 1982. MR 657581 (84h:43027)

[GCK01] J. García-Cuerva and V. Kolyada, *Rearrangement estimates for Fourier transforms in L^p and H^p in terms of moduli of continuity*, Math. Nachr. **228** (2001), 123–144. MR 1845910 (2002j:42014)

[GCRdF85] J. García-Cuerva and J. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland Mathematics Studies, vol. 116, North-Holland Publishing Co., Amsterdam, 1985. MR MR807149 (87d:42023)

[Gra09] L. Grafakos, *Modern Fourier analysis*, second ed., Graduate Texts in Mathematics, vol. 250, Springer, New York, 2009. MR 2463316 (2011d:42001)

[Lat78] R. Latter, *A characterization of $H^p(\mathbf{R}^n)$ in terms of atoms*, Studia Math. **62** (1978), no. 1, 93–101. MR 0482111 (58 #2198)

[LR94] P. Lemarié-Rieusset, *Projecteurs invariants, matrices de dilatation, ondelettes et analyses multi-résolutions*, Rev. Mat. Iberoamericana **10** (1994), no. 2, 283–347. MR 1286477 (95e:42039)

[MSV08] S. Meda, P. Sjögren, and M. Vallarino, *On the H^1-L^1 boundedness of operators*, Proc. Amer. Math. Soc. **136** (2008), no. 8, 2921–2931. MR 2399059 (2009b:42025)

[Pee75] J. Peetre, *On spaces of Triebel-Lizorkin type*, Ark. Mat. **13** (1975), 123–130. MR 0380394 (52 #1294)

[Ste93] E. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993, With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III. MR MR1232192 (95c:42002)

[SW68] E. Stein and G. Weiss, *Generalization of the Cauchy-Riemann equations and representations of the rotation group*, Amer. J. Math. **90** (1968), 163–196. MR 0223492 (36 #6540)

[Tai66] M. Taibleson, *On the theory of Lipschitz spaces of distributions on Euclidean n-space. III. Smoothness and integrability of Fourier tansforms, smoothness of convolution kernels*, J. Math. Mech. **15** (1966), 973–981. MR 0198222 (33 #6381)

[TW80] M. Taibleson and G. Weiss, *The molecular characterization of certain Hardy spaces*, Representation theorems for Hardy spaces, Astérisque, vol. 77, Soc. Math. France, Paris, 1980, pp. 67–149. MR 604370 (83g:42012)